THE OPTIMAL PAYMENT OF CORPORATE INCOME TAX INSTALMENTS

by

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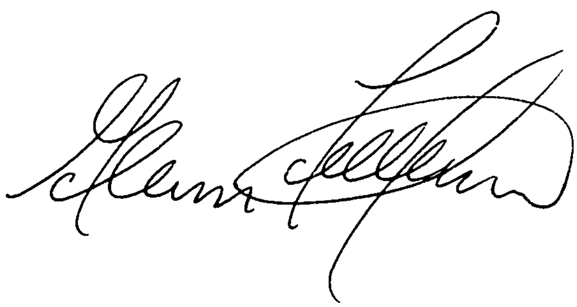

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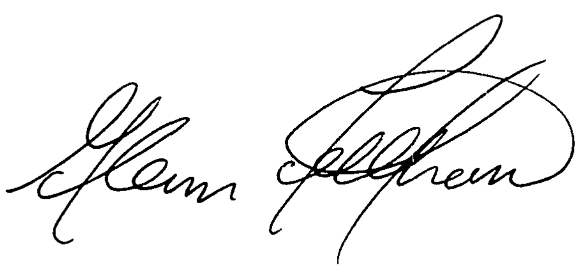

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ABSTRACT

THE OPTIMAL PAYMENT OF CORPORATE INCOME TAX INSTALMENTS

This thesis examines the taxpayer's decision as to the amount and timing of the payment of federal corporate income tax instalments (also known as estimated taxes) for Canadian corporations. The corporation's payment decision is made under uncertainty as instalment payments must normally be paid throughout the year even though an important determinant of the amount owing in instalments is the corporation's tax liability for the year. The problem is formulated as one of dynamic optimization under uncertainty in which that uncertainty is reduced as the financial results for the year become known.

The primary contribution of this thesis is the development of the first theory for any country of the optimal pattern of tax instalment payments. A further contribution, which relates more to the Canadian context, is the mathematical formulation of features of the law which were previously only known through general description and numerical example. As the objective function of the corporation is non-differentiable, analytic optimization methods used previously in static models of decision-making are extended to dynamic optimization under uncertainty. To demonstrate applicability of the results to tax practice, numerical optimization using linear programming is also used to solve the optimization problem. Finally, this thesis contributes to tax policy through the development of a methodology to incorporate instalment structures into marginal effective tax rate analysis and average effective tax rate analysis.

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CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Introductory Comments

A major trend in tax research in accounting is the development of microeconomic theories of decision-making in a world with tax.[1] In contrast to traditional economics-based tax research which has focused on broad aggregates such as savings and investment, this new research is examining micro-oriented decisions for which taxation rules are more likely to be a determining influence, *e.g.*, the timing of transactions, the legal form of transactions, and financial transactions as opposed to real transactions.[2] A problem which is consistent with this new emphasis, and which has yet to be examined formally for any country, is the taxpayer's decision as to the optimal amount and timing of income tax instalment payments. This problem, as it applies to the federal taxation of Canadian corporations, is considered in this thesis.

Studying the payment of corporate income tax instalments is important in that they must be paid by almost all Canadian corporations which have positive tax liabilities.[3] Further, corporate instalment structures are prevalent throughout the world: of 92 countries with a

[1]This trend is reflected in texts by Scholes and Wolfson [1992] and Thornton [1993].

[2]This classification of issues is discussed by Slemrod [1992].

[3]As discussed below, corporations which have zero tax liability in the previous year are generally exempted from paying instalments.

corporate income tax, 82 have instalment systems (pre-payment systems).[4] Countries utilizing instalment structures to collect corporate tax are on every continent, from every type of political system, and at every stage of economic development. The alternative to using an instalment structure, which is to collect tax for the year as a lump sum at the date of the filing of the tax return, is not generally satisfactory to governments as it produces an uneven flow of funds in the year and may increase the levels of tax evasion.

The decision as to the amount to pay in instalments is typically made under uncertainty, as instalment payments must normally be paid throughout the year even though an important determinant of the amount owing in instalments is the tax liability for the entire year. For example, in Canada corporations must make their first instalment payment at the end of the first month of the fiscal year; as only one-twelfth of the year is completed, the tax liability for the entire fiscal year is uncertain. Hence taxpayers have a two-part problem: first, to form beliefs about their distribution of the tax liability for the year; and second, given their distributional beliefs, to determine the optimal amount of instalment payments given the

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[4]In reviewing the Worldwide Corporate Tax Guide (Ernst and Young [1992]) and Tax Policy in OECD Countries (Messere [1993]), the following countries have been identified as having corporate instalment structures: Angola, Argentina, Australia, Austria, Bangladesh, Barbados, Belgium, Bolivia, Brazil, Canada, Chile, China, Columbia, Costa Rica, Cyprus, Djibouti, Dominican Republic, Ecuador, Egypt, Fiji, Finland, France, Gabon, Germany, Gibraltar, Greece, Guam, Honduras, Hong Kong, Hungary, Iceland, India, Indonesia, Ireland, Italy, Ivory Coast, Jamaica, Japan, Jordan, Kenya, Kuwait, Libya, Luxembourg, Malaysia, Mauritius, Mexico, Morocco, Mozambique, Namibia, Netherlands, New Zealand, Nigeria, Norway, Panama, Papua New Guinea, Paraguay, Peru, Philippines, Poland, Portugal, Puerto Rico, Qatar, Russian Federation, Senegal, Singapore, South Africa, Spain, Sri Lanka, Surinam, Swaziland, Sweden, Taiwan, Tanzia, Thailand, Trinidad and Tobago, Turkey, Uganda, the United States, and Uruguay. Countries identified as not having corporate instalment structures are: Botswana, Denmark, El Salvador, Guatemala, Iran, the Isle of Mann, Korea, Saudia Arabia, the United Kingdom, and Zimbabwe.

consequences of paying too little ("underpayment") and paying too much ("overpayment").[5] This thesis develops a theory to explain the solution to the second part of this problem.

The corporation's problem in this thesis is considered in a decision-theoretic setting; that is, government rules concerning instalment structures are exogenous to the models. If the government was modelled as an active player, the problem would be one of optimal taxation -- the government would choose an instalment structure to maximize a social welfare function subject to the corporation making optimal payments.[6] This approach is not used in this thesis as the instalment structure which applies in Canada would emerge as optimal only if a tremendous amount of structure were placed on the form of the social welfare function -- if indeed it would emerge as optimal at all. Further, the rules concerning the instalment structure are fixed in law and are only subject to change through an act of Parliament. Hence, at least in the short run, it is appropriate to consider these rules as being fixed and unresponsive to taxpayer behaviour.

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[5]These terms are used here in a general sense; more detailed modelling of the consequences of alternative payment amounts is provided in chapter 2 below.

[6]See Auerbach [1985] for a survey of the optimal taxation literature.

1.2 Review of the Literature

Most of the literature on instalment payments has been written by tax practitioners. This literature has produced scattered insightful comments, each of which focuses on specific aspects of the instalment rules. There is as yet no comprehensive theory.

The most common practitioner insight is based on the fact that a Canadian corporation's instalment liability is a function of a corporation's tax liability for the current year and for the preceding two years. It is argued that the taxpayer should pay an amount based on his or her tax liability for the preceding two years if its tax liability for the current year is expected to be greater than in the past (*e.g.*, Scheuermann [1988,10:11]). The results of this thesis show that this insight is correct where the corporation knows its tax liability for the year with certainty. However, if there is uncertainty, it is only correct if there is zero probability that the tax liability could decrease.

Where tax liability for the year is expected to be greater than in the past, there have been two alternative suggestions for optimal payment strategy. Scheuermann [1988,10:11] appears to suggest that in this situation the instalment payment should be based on the expected value of the current year's tax liability:

> Where tax liability is expected to decrease, however, tax instalments are normally made on the basis of estimated tax liability for the current year in order to improve the taxpayer's cash flow position.

The second suggestion, by Carr and Yull [1994,35], is that taxpayers should balance the cost of underpayment and the cost of overpayment in arriving at an optimal payment amount:

> Taxpayers who are required to make ... instalment payments often find themselves paying either too much or too little. Overpayments are undesirable since they constitute interest-free loans to Revenue Canada; underpayments, on the other hand, result in nondeductible interest charges and possible penalties.

The findings of this thesis suggest that the latter view is closer to being correct. However, Carr and Yull have not correctly assessed the cost of underpayment: the opportunity loss from making an interest-free loan to Revenue Canada applies to underpayments as well as to overpayments.

Each of the above insights is concerned only with the total amount of instalment payments. They do not consider the optimal timing of payments within the year. This subject has been barely considered in the literature. Price Waterhouse [1992,12] makes the following comment:

> The interest offset method essentially permits a corporation to make catch-up payments when it discovers it has not made sufficient payments. ... Care must be exercised in calculating just how much extra the corporation should pay under the interest offset method.

A final insight concerns the situation in which a taxpayer's opportunity cost of funds is so high that it is optimal to "borrow" from the government by deliberate underpayment of instalments. For example, Stark [1991,1415-1416] comments: "In 1975, the six-percent deficiency interest rate and the absence of compounding encouraged taxpayers to make the IRS the lender of choice." This type of strategy has also been considered in the academic literature. Feltham and Paquette [1992] consider the conditions under which it would be optimal for a firm to borrow from the government, and the effects of this borrowing on the firm's capital structure; particularly in light of the priority of the government's claim on bankruptcy. Since governments generally set interest rates on underpayments such that such deliberate underpayment is optimal only for taxpayers facing a very high cost of funds in the private lending market, in this thesis it is assumed for the most part that parameter values are such that deliberate underpayments are not optimal.[7]

As mentioned above, there are no previous theories in the academic literature on the optimal payment of instalments apart from the models of optimal borrowing discussed above. There are some empirical papers including experimental economics studies. Therefore, one of the contributions of this thesis is to provide theoretical guidance to the empirical literature. As noted by Udell [1991,48];

> What I feel is missing ... is an incentive theory about why taxpayers should choose to be in any of ... these categories [to underpay or overpay]. The one thing that a system of penalties provides is a set of exogenous incentives for behavior.

One of the earlier empirical papers is by Christian, Gupta, and Willis [1992], who examine the characteristics of taxpayers with underpayments and overpayments. This was extended by Christian, Gupta, Weber, and Willis [1994] to include a role for tax preparers. Other papers which have empirically examined the effect on taxpayer behaviour of being in a balance due position versus a refund position include Chang, Nichols, and Schultz [1987], Hite, Jackson, and Spicer [1988], Chang and Schultz [1990], Schepanski and Kelsey [1990], Martinez-Varzquez, Harwood, and Larkins [1992], and White, Harrison, and Harrell [1993]. Moore, Steece, and Swenson [1985], in examining the effect on instalment payments of the passage of Proposition 13 in California, found that an increase in income tax liability (through a reduced deduction for property taxes), provided an immediate and permanent increase in aggregate personal and corporate income tax instalments.

The academic literature on tax compliance, although related, has a significantly

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[7]Specifically, it is assumed that $G_i > C_i$. See chapter 2 for definitions of these parameters.

different focus. In that literature, taxpayers determine filing positions and then have some probability of being audited and re-assessed by the tax authority. The taxpayer does not know if he or she will be audited and hence makes the tax filing decision under uncertainty. The primary focus of the existing compliance literature is therefore an examination of the effects on filing behaviour of asymmetric information between the taxpayer and the taxing agency[8]. Hence, the existing compliance literature deals with the effect of auditing by the revenue authority on the amount that the taxpayer chooses to declare as his or her tax liability. This thesis is fundamentally different from that literature, in that in this thesis the decision in question is the optimal timing of instalment payments in respect of that liability; it is assumed that the taxpayer will ultimately pay the true amount of tax liability owing. A possible extension of this research is the inclusion of both optimal instalment payments and optimal tax filing positions. This is discussed in the conclusion of this thesis.

[8]The following is a sample of papers which fall within that basic structure: Beck and Jung [1989], Graetz, Reinganum, and Wilde [1986], and Reinganum and Wilde [1986]. Roth, Scholtz, and Witte [1989], and Roth and Scholtz [1989] provide a general review of the tax compliance literature.

1.3 Thesis Outline

Chapter 2 develops an expression for the corporation's loss for any given time series of instalment payments and tax liability for the year. In formulating this loss, provisions of the Income Tax Act, and the corporation's opportunity losses and gains, are modelled.

The loss expression from chapter 2 is used in chapters 3 and 4 to develop, for a risk neutral corporation, the expected loss for any time path of instalment payments, which is the objective function to be minimized by the corporation. A primary difference between chapter 3 and chapter 4 is in the assumption regarding the distribution of the corporation's tax liability for the year: in chapter 3 the distribution is assumed to be continuous while in chapter 4 it is assumed to be discrete. As there are analytic complexities associated with multi-period continuous distributions, the models in chapter 3 utilize a single payment setting. Chapter 4 captures more of the institutional detail in allowing for monthly instalment payments and for a more sophisticated treatment of uncertainty. In chapter 4, uncertainty is presented as evolving over time; for example, in the last month of the fiscal year, a corporation has better information about its liability for the year than it did in the first month of the year.[9]

Chapter 3, which develops the single-payment model, begins by analyzing an instalment structure in which there is a constant marginal cost of both underpayment and overpayment. The results obtained are similar to those for the "newsvendor"[10] problem in management

[9]Limberg [1987] provides an introduction to tax decision-making under uncertainty. Huddart [forthcoming] considers the evolution of uncertainty over time in connection with the optimal exercise of employee stock options.

[10]Scholes and Wolfson [1992,176] discuss another tax planning problem which may be modelled as a newsvendor problem.

science, except that there exists a barrier created by the use of previous years' tax liability in defining an underpayment. The second part of the chapter extends the model to include the Canadian 50% interest penalty on substantial underpayments which was introduced in 1989. It is shown that the corporation's optimum may occur at a kink in the objective function where the corporation is just becoming subject to the 50% interest penalty.

As discussed above, chapter 4 solves the corporation's instalment payment problem under more realistic assumptions. Note, however, that this chapter assumes that the corporation has perfect foresight concerning interest rates in the year and that interest rates are unchanging in the year. In section 4.4, it is further assumed that interest rates are simple (are not compounded) and that the corporation's tax refund, if any, is paid by the government on the remainder due date for the fiscal year. Under these assumptions, it is shown that the corporation's optimal strategy is to make no instalment payments before the last month of the fiscal year. In sections 4.5 and 4.6, further analytic results are derived where tax liability for the year is known with certainty. In particular, it is shown that compound interest rates and allowing the government to delay payment of refunds create incentives for the corporation to pay earlier in the year.

Since the model in chapter 4 is more general than that in chapter 3, the results in chapter 3 are in most respects special cases of the results in chapter 4. However, the use of a continuous distribution for the tax liability does allow for some intuition into the problem which is not available with a model utilizing a discrete distribution. Chapter 3 may be omitted by the reader if the greater intuition into the problem is not desired.

A special method is used in chapters 3 and 4 to determine optimal instalment payment strategies as the objective function is non-differentiable. This non-differentiability arises as the modelling of the income tax legislation requires the use of maximization and minimization operators in the definitions of key problem elements such as the 50% penalty. Diewert [1981] describes a useful analytic optimality condition for such problems which has been further developed by Macnaughton [1993]. Chapters 3 and 4 use these Diewert conditions in determining the optimal instalment payment strategy for the corporation.

Chapter 5 is aimed at making the analytic model of chapter 4 useful to practitioners. Accordingly, the chapter first provides a linear programming formulation of the expected loss function presented in chapter 4. The second section of the chapter, in abandoning theoretical purity, discusses a practical method for implementing the linear program where the corporation updates the model throughout the year to deal with errors in the forecasts in the interest rates used in the optimization. The final section of the chapter discusses a more theoretically correct treatment of interest rates; treating interest rates as a second stochastic variable (in addition to the corporation's tax liability for the year). However, for technical reasons dealing with the size of this problem, it is unlikely that it would be used by tax practitioners.

Chapter 6 focuses on the development of methods to examine policy implications of the instalment structure. A measure of the percentage difference in tax from a benchmark structure is developed. This is in the nature of an effective tax rate measure. Potential applications of this method are then illustrated through an example.

Finally, conclusions and directions for future research are set out in chapter 7.

1.4 Contributions

A brief summary of the contributions of this thesis is provided in this section. The primary contribution is the development of the first theory of the optimal pattern of tax instalment payments for any country. Although instalment structures are a basic feature of corporate income tax systems throughout the world (as well as personal income tax systems as they relate to the self-employed), all previous academic work in this area has been empirical. The need for theory to guide the empirical testing has been noted in the literature. Tax practitioners should also be interested in the theory development as the practitioner literature provides only scattered insights as to how a taxpayer should pay instalments.

Although the theory is developed under Canadian tax rules, the three key features of the model generalize readily to other countries. First, in all countries with instalment structures, corporations choose optimal payment amounts through trading off tax and non-tax factors. Overpayments result in opportunity costs, while underpayments result in lowered opportunity costs but increased tax costs through government-imposed interest charges. Second, in about one-half of the countries with instalment structures, including Canada, the corporation must choose payment amounts under uncertainty. As noted above, this uncertainty arises as the corporation's instalment liability is determined (in part) by the corporation's tax liability for the fiscal year, which is generally unknown at the time instalments are paid. Third, in almost all countries with instalment structures, corporations must make multiple instalment payments in the tax year. Therefore, the appropriate modelling framework is a dynamic uncertainty formulation in which uncertainty is reduced over time as the financial results for the year become known.

Other contributions of this thesis relate more specifically to its Canadian context. The principal Canada-specific contribution of the model formulation is the rigorous mathematical formulation of features of the tax law which have previously been known only through general description and through numerical examples. In particular, the mathematical formulations for instalment liability and instalment interest provides more thorough analysis of the logic of the law than was previously available.

A further contribution of this thesis is the solution of the model. Because the objective function of the corporation is non-differentiable, a special method for finding an analytic solution is required. Previous work on this type of problem considers only static optimization problems under conditions of certainty. This thesis shows that this method can be applied to more complex problems involving dynamic optimization and/or optimization under uncertainty. This optimization method should find use in many tax planning problems, both in the area of instalments and elsewhere.

Since a use of the model developed in this thesis is to aid tax practice, methods for numerically solving the model using linear programming are also investigated. A second contribution in this area is the investigation of the tradeoffs between greater model realism and model size, particularly concerning stochastic interest rates and the frequency with which new information arrives during the fiscal period.

Finally, the thesis contributes to tax policy through the development of a methodology to incorporate instalment structures into marginal effective tax rate analysis and average effective tax rate analysis. Marginal effective tax rate formulations have been criticized for incorporating only selected features of the tax system; this work partially addresses this concern. The analysis of average effective tax rates illustrates the inequities associated with a tax system which penalizes sub-optimal tax planning decisions.

CHAPTER 2

THE LOSS FROM INSTALMENTS

This chapter develops the corporation's loss from instalments. In further chapters, optimal payment strategies are examined which minimize this loss or, where tax liability for the year is uncertain, the expected value of this loss. In section 2.1 the timing of payments and liabilities in the instalment process is discussed. The corporation's loss from any time path of instalment payments is then developed in section 2.2. An alternative formulation, which provides identical optimal payment strategies, is then provided in section 2.3. Under that formulation, the present value as of the start of a fiscal year of all payments required by the Income Tax Act to be paid by the taxpayer to Revenue Canada in respect of that year is determined (where refunds by Revenue Canada are included as negative payments).

2.1 The Timing of Payments and Liabilities in the Instalment Process

The Income Tax Act stipulates that corporations shall make instalment payments on or before the last day of each month.[11] If the corporation's tax liability for the fiscal year[12]

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[11]Subsection 157(1). This analysis does not cover provincial corporate income taxes of Ontario, Quebec, and Alberta, as these provinces, in opting-out of the tax collection agreement, set up their own instalment structures.

[12]The phrases "tax liability" or "tax liability for the year" will be used to refer to the corporation's tax payable under Parts I, I.3, VI and VI.1 for the year; the amount specified in subparagraph 157(1)(a)(i).

is greater than the total amount of the 12 monthly instalment payments, the excess is then due at the end of the second or third month[13] following the end of the year (hereafter the "remainder due date"). Alternatively, if the sum of the instalment payments exceeds the corporation's tax liability for the year, the excess is refunded at some date after the remainder due date (after Revenue Canada has assessed the corporation's tax return). To simplify analysis, it is assumed that this refund date is known with certainty. In this thesis, the period from the first day of the corporation's tax year to the remainder due date is referred to as the "instalment period" and the period from the remainder due date to the date of refund is referred to as the "stub period". The following time line sets out the payment and refund dates, and the instalment and stub periods. The payment amounts at these dates constitute the complete set of endogenous variables in this thesis.

[13]Under paragraph 157(1)(b), the remainder will be payable two months after the end of the taxation year, unless the corporation was a Canadian-controlled private corporation which deducted an amount for the small business deduction in the current or preceding year, in which case the remainder is generally payable three months after the end of the taxation year.

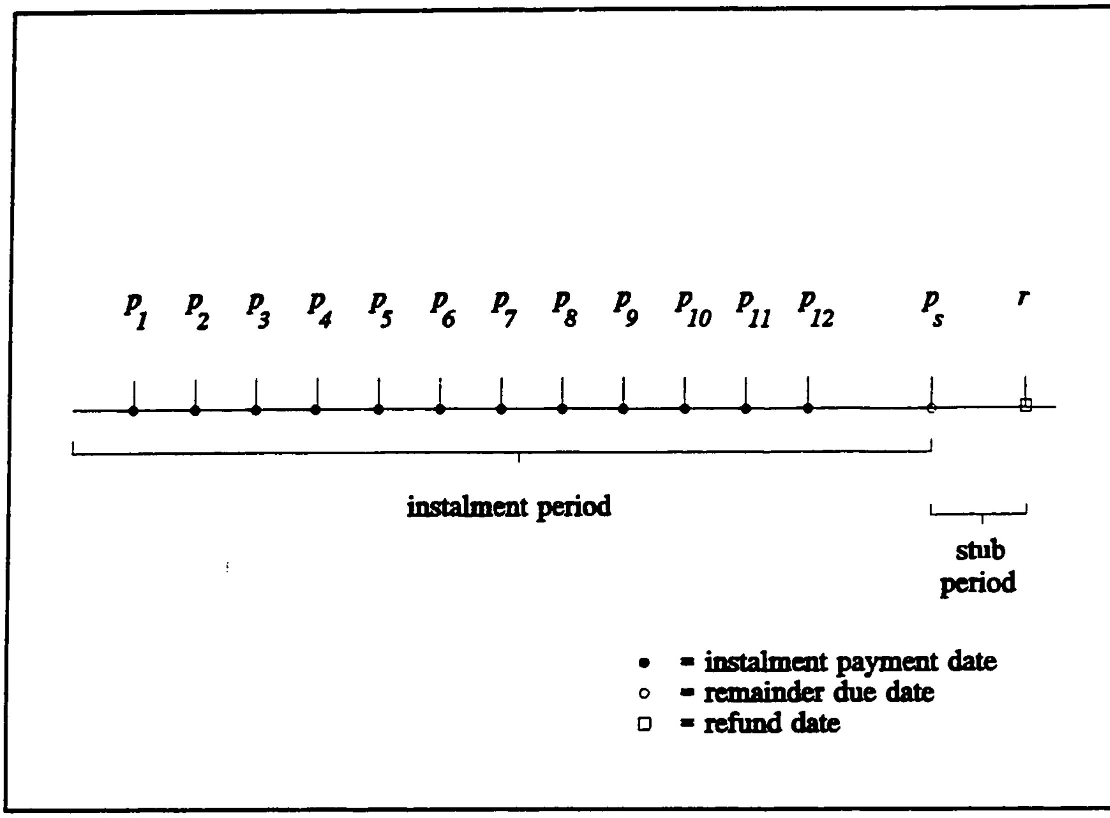

FIGURE 2.1
The Payment Structure

Throughout this thesis the twelve monthly payments, made on the last day of each month, is denoted p_1 through p_{12} respectively. The stub payment, made on the remainder due date, is denoted p_s. An assumption has been made that the payments occur only at these dates to simplify analysis. The refund amount (a payment from Revenue Canada), is denoted r. The vector of payments p_1 through p_{12} is denoted as the vector p throughout this thesis.

2.2 Determining the Corporation's Loss

For any tax liability for the year, a taxpayer's loss from the payment of instalments may be defined as the aggregate of four amounts;[14]

$$l = U + O + Pen + Stub, \tag{2.1}$$

where U is the amount of interest owing under section 161 from underpaying in the instalment period ("instalment interest"[15]), O is the opportunity loss or gain through overpaying or underpaying in the instalment period, Pen is the penalty under section 163.1 associated with underpayment in the instalment period, and $Stub$ is the opportunity loss from overpaying in the year such that the taxpayer will receive a refund at the end of the stub period. The four amounts U, O, Pen, and $Stub$, are developed in sections 2.2.1, 2.2.2, 2.2.3, and 2.2.4 respectively. Note that each of the four amounts is a function of the endogenous variables listed above; *i.e.*, the monthly payments p_1 though p_{12}, and the stub payment p_s.

The focal date or comparison date for determining each of these four amounts above is the remainder due date. That is, all losses are taken forward or discounted to that date. Note that alternative focal dates, such as the beginning of the tax year, could have been chosen (the first date of the fiscal year is used as the focal date in section 2.3). To determine the value of the loss or the expected loss at any alternative date, the loss or expected loss may simply be discounted or taken forward to that alternative focal date.

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[14]This analysis does not apply to the capital tax imposed on financial institutions and on large corporations which have a low tax liability (the LCT). See sections 181.7 and 190.21 to 190.23 of the Income Tax Act.

[15]Revenue Canada, in Information Circular 81-11R3, Paragraph 10, uses the term "instalment interest" to describe this amount.

2.2.1 Interest Owing under Section 161 from Underpaying During the Instalment Period

As stated above, the Income Tax Act stipulates that corporations shall make monthly instalment payments. Where these payments are not adequate (as defined below), the corporation will owe interest to Revenue Canada. The determination of the amount of interest owing under section 161 from underpaying in the instalment period, U, for any series of monthly instalment payments (*i.e.*, for any time path of payments), is described in this section. As the determination of U is complex, it is modelled in two steps. First, instalment liability is defined. Second, the amount of interest owing from underpaying in the instalment period, U, is constructed from the corporation's instalment liability and its instalment payments.

Instalment Liability

Subsection 157(1) provides three alternative methods for calculating instalment payments. These three methods are as follows:[16]

Method *I*. pay each month an instalment of 1/12 of the estimated tax payable;

Method *II*. pay each month an amount equal to 1/12 of the corporation's first instalment base;[17] and

Method *III*. pay each of the first 2 months an amount equal to 1/12 of the second instalment base, and for the remaining 10 months pay an amount equal to 1/10 of the remainder of the first instalment base (that is, after deducting the first 2 instalments from the first instalment base).

Subsection 161(4.1) provides that the corporation is liable to pay instalments according to whichever of the three methods generates the least total amount of instalments for the year.

[16]One minor feature of the instalment system which is not modelled is that the amount calculated using Method *I* equals zero if tax liability for the year is $1,000 or less for most corporations and $10,000 or less for credit unions. See subsections 157(2) and 157(2.1) of the Income Tax Act.

[17]The first and second instalment bases, which are defined in Regulation 5301(1) and (2), are the corporation's tax liability for the immediately preceding year and the second preceding year respectively. These instalment bases are not affected by the application of future years' loss to reduce taxable income. Special rules apply to corporations which have amalgamated with other corporations, and to corporations with taxation years of less than 12 months.

The total amount of instalments for the fiscal year under Method *I* is,

$$12x \tag{2.2}$$

where x is 1/12 of the corporation's tax liability for the year.[18] Similarly, the total amount of instalments for the fiscal year under Method *II* is,

$$12b_1 \tag{2.3}$$

where b_1 is 1/12 of the corporation's first instalment base for the year.

The calculation of the total amount of instalments for the fiscal year under Method *III* is more complex. In each of the first two months of the year the corporation pays an amount b_2, where b_2 is 1/12 of the corporation's second instalment base for the year. In each of the remaining 10 months the corporation will pay 1/10 of the amount, *if any*, by which the first instalment base ($12b_1$) exceeds the amount paid under the first two instalments ($2b_2$). Hence, the total amount of instalments for the fiscal year under Method *III* is,

$$\begin{cases} 2b_2 + (10)\left(\frac{1}{10}\right)(12b_1 - 2b_2) = 12b_1 & \textit{if } 2b_2 \leq 12b_1 \\ 2b_2 & \textit{otherwise} \end{cases} \tag{2.4}$$

A comparison of expressions (2.2), (2.3), and (2.4) above shows that the rule in subsection 161(4.1) that the corporation is liable to pay instalments under the method which generates the least total amount of instalments for the year implies that the applicable method is determined

[18]Tax liability is treated as exogenous in this thesis. Future extensions in which tax liability is endogenous are discussed in section 7.2, Directions for Future Research.

as follows:

Method *I* (total payments $12x$) if $x \leq b_1$;

Method *II* (total payments $12b_1$) if $x \geq b_1$ and $2b_2 \geq 12b_1$; and

Method *II* or Method *III* (total payments $12b_1$) if $x \geq b_1$ and $2b_2 \leq 12b_1$.

In the third category above, there is an ambiguity as to whether Method *II* or Method *III* is to be applied. However, there is a rule of interpretation of statutes that if a provision is present in law, it is assumed to have purpose. If Method *II* was always chosen in this third category, Method *III* would not apply in any circumstance; hence, by this rule of statutory interpretation, Method *III* must be selected for at least some parameter values. Furthermore, Method *III* is more favourable to the taxpayer where $b_2 < b_1$: although both methods result in the same total instalments in the fiscal year, where $b_2 < b_1$ these payments have a lower present value as amounts are paid later in the year. As it is a rule of statutory interpretation that the resolution of ambiguity is to favour the taxpayer, Method *III* should apply in this situation. Therefore, for both of these reasons, Method *III* is the appropriate method in the third category above if $b_2 < b_1$. Thus, the choice of methods can be restated as follows:

Method I ***if*** $x \leq b_1$ **(2.5)**

Method II ***if*** $b_1 \leq x$ ***and*** $b_1 \leq b_2$ **(2.6)**

Method III ***if*** $b_2 \leq b_1 \leq x$ **(2.7)**

Hence, the corporation's instalment liability at each payment date during the fiscal year is,

$$q_i = \begin{cases} x & \forall\ i = 1 \text{ to } 12 & \text{if } x \le b_1 \\ b_1 & \forall\ i = 1 \text{ to } 12 & \text{if } b_1 \le \{x, b_2\} \\ \begin{cases} b_2 & \forall\ i = 1,2 \\ \frac{1}{10}(12b_1 - 2b_2) & \forall\ i = 3 \text{ to } 12 \end{cases} & & \text{if } b_2 \le b_1 \le x \end{cases} \tag{2.8}$$

The definition of instalment liability presented above is based on an amendment to subsection 161(4.1) of the Income Tax Act which received Royal Assent on June 15, 1994. Since this is a very new provision, there has been no official interpretation of this provision by Revenue Canada. Hence, there remains a possibility that Revenue Canada will interpret the provision in a way which is more favourable to the corporation than the analysis given above.

Prior to the effective date of this amendment (*i.e.*, for taxation years before 1992), the definition of instalment liability was significantly different. An analysis of this prior definition is given in Appendix A. The appendix demonstrates that the prior law allowed for switching among methods in the course of a fiscal year. The analysis in appendix A is quite different from that given in prior literature and may be of interest to corporations as a basis for asking for a refund of excess instalment interest charged by Revenue Canada.

Instalment Interest

Where a corporation has paid less than the instalment liability at any payment date, under subsection 161(2) the taxpayer will owe interest on the unpaid amount.[19] Any interest owing from underpaying instalments in the instalment period may however be offset through overpaying other instalments in the period by virtue of subsection 161(2.2). That is, interest to the taxpayer from overpayment will reduce (in the limit to zero), the amount of interest payable.

The offset-interest provision, subsection 161(2.2), states that the amount of instalment interest payable by the corporation, U, is the amount as determined in that subsection but "shall not exceed" the amount payable under subsection 161(2). Appendix B demonstrates that the "shall not exceed" requirement is unnecessary, and therefore instalment interest is completely defined by subsection 161(2.2).

The amount set out in subsection 161(2.2) is the amount, if any, by which paragraph 161(2.2)(c) exceeds paragraph 161(2.2)(d). Paragraph (c) is the amount of interest that would be payable under 161(2) if no instalments were paid. Paragraph 161(2) requires that the corporation pays interest on an amount that the taxpayer "failed to pay ... on or before the date the amount was required to be paid". The amount that the taxpayer "failed to pay" is thus the entire instalment liability. Hence, paragraph 161(2.2)(c) may be written as,

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[19]This interest is deemed to be zero if it is $25 or less (subsection 161(2.1)). This feature is not considered in the model.

$$\sum_{i=1}^{12} q_i g_i \tag{2.9}$$

where q_i is defined in equation (2.8) above and g_i is the amount of instalment interest owing by the corporation for a deficiency of payment of $1 arising at payment date i.

Subsection 248(11) requires that instalment interest be compounded daily. Therefore,

$$g_i = \left[\prod_{k=i+1}^{13} \left(1 + \frac{G_k}{365}\right)^{N_k}\right] - 1 \tag{2.10}$$

where G_k is the prescribed rate of interest in period k and N_k is the number of days in period k. Periods 2 through 12 are the 11 months of the fiscal year which follow the first payment date. Period 13 is the time between the last day of the fiscal year and the remainder due date. The prescribed rate of interest is defined in Regulation 4301 to be essentially the quarterly adjusted weekly average rate on 90 day treasury bills plus 2 percentage points. Further details on the prescribed rate are given in chapter 5.

Paragraph 161(2.2)(d) is the amount of interest that would be paid to the corporation under subsection 164(3) if it were applied to the instalment period and no tax was payable by the corporation for the year. The amount, which is also compounded by virtue of subsection

248(11), may be written,

$$\sum_{i=1}^{12} p_i g_i \tag{2.11}$$

where p_i is the amount paid at time i.

The amount determined under 161(2.2) is therefore the amount, if any, by which paragraph 161(2.2)(c) (equation (2.9) above) exceeds paragraph 161(2.2)(d) (equation (2.11)), or[20]

$$U = \max\left[0, \sum_{i=1}^{12} (q_i - p_i) g_i\right] \tag{2.12}$$

The amount of interest from underpayment in the instalment period, U, is therefore a function of past and present tax liability (as q is a function of x, b_1, and b_2), payments p_1 through p_{12}, and the rates g_i. Note that the words "if any" require the use of the maximization operator.

2.2.2 The Opportunity Loss or Gain though Overpaying or Underpaying During the Instalment Period

The preceding section defined instalment interest, U, the first component of the corporation's loss in equation (2.1) above. The second element in determining a corporation's loss from the payment of instalments, $l(p,p_s;x)$, is the opportunity loss or gain through overpaying or underpaying in the instalment period, O.

The corporation's opportunity loss or gain arising from any payment i is the difference

[20]Appendix B demonstrates that this method of calculating instalment interest is equivalent to the method used by Revenue Canada in Information Circular 81-11R3.

between its payment amount p_i and its instalment liability for that month q_i, multiplied by the corporation's after-tax cost of capital compounded over the period from the payment date to the remainder due date c_i,

$$(p_i - q_i)\, c_i \tag{2.13}$$

where

$$c_i = \left[\prod_{k=i+1}^{13} \left(1 + \frac{C_k}{365}\right)^{N_k} \right] - 1 \tag{2.14}$$

The rate C_k is the corporation's after-tax cost of capital in period k expressed as an annual rate of interest. Assuming that the corporation is increasing borrowing or decreasing debt-type savings to make the tax payments,[21] the corporation's after-tax cost of capital at time i, c_i, is the interest rate on that borrowing or savings multiplied by one minus the corporation's marginal tax rate.[22] [23] Recall that the focal date or comparison date for determining the

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[21]It is assumed that the corporation is able to borrow any amount at the fixed rate of interest. For a model of tax compliance with borrowing constraints, see Andreoni [1992]. Miles [1967] presents reasons why tax instalments are likely to be financed by short-term debt.

[22]The taxpayer's marginal tax rate enters into the calculation because interest income is generally fully taxable, and the taxpayer can in practice (although not in law) deduct interest on funds borrowed to pay instalments. As stated by Couzin [1991,79]:

> [a]s for Revenue practise, no one asks awkward questions such as whether corporate borrowing levels under lines of credit may have increased the day a tax instalment was paid, or whether money has been borrowed to pay interest.

[23]Note that the rate C_k, which is an after-tax rate, is affected by the corporation's tax liability for the year, X (where $X = 12x$), to the extent the corporation's marginal tax rate is affected by X. C_k may further be affected by X as the firm's profitability, which will normally be positively correlated with X, may affect its cost of capital. Although C_k may therefore be a function of X, to simplify analysis it is treated as an exogenous variable.

corporation's loss or expected loss is the remainder due date. The corporation's after-tax cost of capital is compounded daily as this would appear to best reflect reality.[24] For example, loans from financial institutions typically use daily compounding.

Summing expression (2.13) over the 12 monthly payments in the instalment period, the corporation's opportunity loss or gain in the instalment period is,

$$O = \sum_{i=1}^{12} (p_i - q_i)\, c_i \tag{2.15}$$

[24]Note that alternative compounding assumptions in determining the corporation's cost of capital will not affect the analytic results in this thesis, although they may marginally affect the corporation's loss.

2.2.3 Penalty

Where a corporation has interest owing under section 161, it may be subject to a penalty. Since 1990 the Income Tax Act has imposed a penalty under section 163.1 for underpayment of instalments equal to 50% of the amount, if any, by which instalment interest payable in respect of instalments as of the remainder due date exceeds the greater of (*i*) \$1,000, and (*ii*) 25% of the interest that would have been payable for the year if no instalment payments had been made for the year. This penalty is therefore in the nature of a penalty on substantial underpayments; in particular, if the interest on deficient instalments does not exceed \$1,000, there is no penalty. The penalty may be written;

$$Pen = .50 \cdot \max\left[0,\ U - \max\left(1000,\ .25 \sum_{i=1}^{12} q_i g_i\right)\right] \tag{2.16}$$

Substituting U from equation (2.12) into equation (2.16),

$$Pen = .50 \cdot \max\left[0,\ \max\left(0,\ \sum_{i=1}^{12} (q_i - p_i) g_i\right) - \max\left(1000,\ .25 \sum_{i=1}^{12} q_i g_i\right)\right] \tag{2.17}$$

As the second maximization operator is redundant, equation (2.17) may be rewritten,

$$Pen = .50 \cdot \max\left[0,\ \sum_{i=1}^{12} (q_i - p_i) g_i - \max\left(1000,\ .25 \sum_{i=1}^{12} q_i g_i\right)\right] \tag{2.18}$$

2.2.4 Opportunity Loss from Waiting for a Refund

At the remainder due date a corporation will either have paid less in the year in instalments than its tax liability for the year, or have paid more than its tax liability for the year. If the corporation has paid less than its tax liability for the year (which it is assumed to know with certainty at the remainder due date), it will pay that deficient amount on the remainder due date in order that it does not incur any further losses through interest charges under subsection 161(1). Otherwise, it will pay zero. That is, the corporation will make the following payment at the remainder due date,

$$p_s = \max\left(0, \sum_{i=1}^{12} (x - p_i)\right) \tag{2.19}$$

Conversely, if the corporation's payments are greater than its tax liability for the year, it will receive a refund of $\sum_{i=1}^{12} (p_i - x)$ at some future date.[25] To simplify analysis, it is assumed in this thesis that this refund date, r, is known with certainty. As the corporation does not have the use of this amount for the period between the remainder due date and the refund date (the "stub" period), an opportunity loss will arise. However, the corporation may receive interest on this overpayment amount, as set out in subsection 164(3), at the prescribed rate. Interest does not accrue to corporations until at least 120 days after the end of the year (paragraph 164(3)(b)), and ends when the amount is refunded, repaid, or applied. Note that

[25]The refund is smaller than indicated by this formula if the corporation owes Part IV tax or other elements of corporate income tax for which instalments are not required.

interest paid to the taxpayer on overpayments is taxable.

Let us view the consequences following the instalment period as being composed of two potential effects: first, the opportunity loss from not having the use of the overpayment amount in the stub period; and second, interest paid to the corporation from the tax authority for the period, if any, which the refund date is after the date defined in paragraph 164(3)(b).

The opportunity loss may be written as follows,

$$\frac{\max\left[0,\sum_{i=1}^{12}(p_i - x)\right] \cdot \left[\left(1 + \frac{C_s}{365}\right)^{N_s} - 1\right]}{\left(1 + \frac{C_s}{365}\right)^{N_s}} \tag{2.20}$$

where N_s is the number of days between the remainder due date, s, and the refund date, r. The numerator in equation (2.20) represents the opportunity cost to the corporation at the refund date of having paid an amount greater than the corporation's tax liability in the instalment period. The denominator discounts this amount to the remainder due date (the focal date).

The gain from interest paid by the government is,

$$\frac{\max\left[0,\sum_{i=1}^{12}(p_i - x)\right] \cdot \left[\left(1 + \frac{G_z}{365}\right)^{N_z} - 1\right]}{\left(1 + \frac{C_s}{365}\right)^{N_s}} \tag{2.21}$$

where N_z is the number of days from the date z at which interest accrues to the corporation which is usually 120 days after the corporation's year end, and the refund date. The numerator in equation (2.21) represents the interest payable by Revenue Canada as of the refund date. The denominator discounts this amount to the remainder due date.

Diagrammatically, these may be represented as follows:

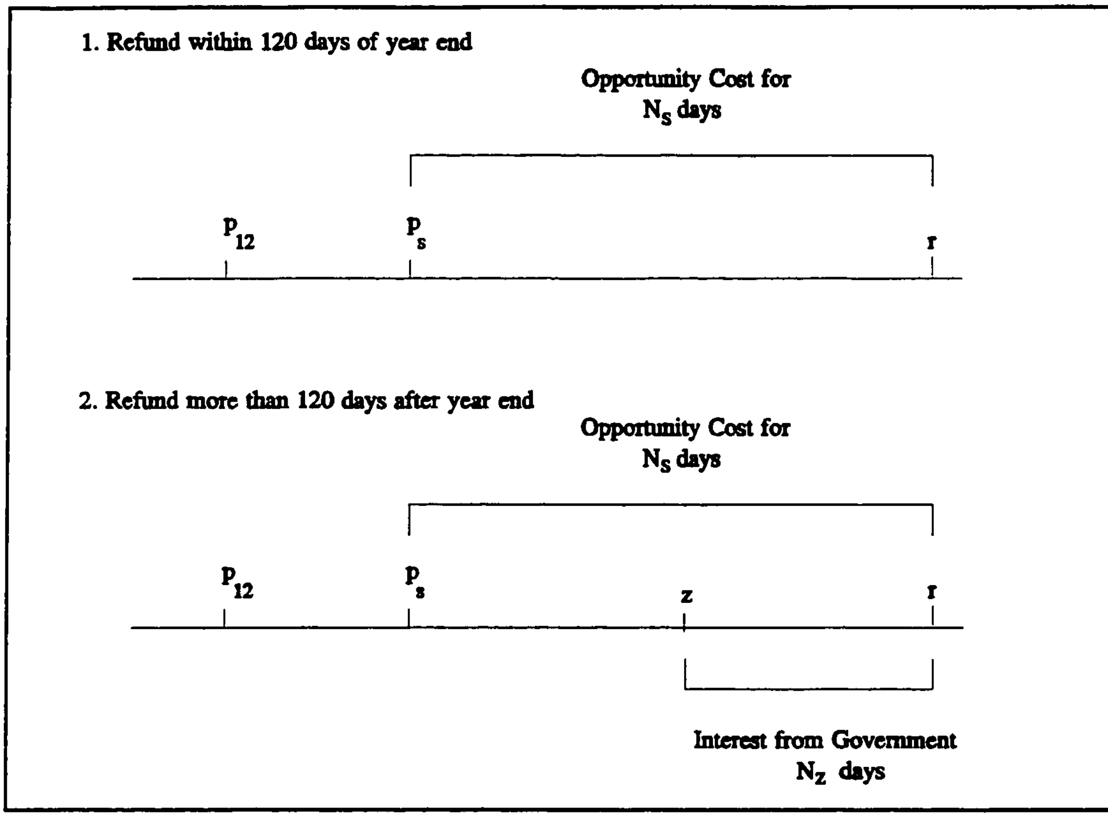

FIGURE 2.2

The Stub Loss

The total loss for the stub period therefore equals the opportunity loss in equation (2.20) less the interest gain in equation (2.21), or,

$$Stub = \frac{\max\left[0,\sum_{i=1}^{12}(p_i - x)\right]\cdot\left[\left(1+\frac{C_s}{365}\right)^{N_s} - \left(1+\frac{G_z}{365}\right)^{N_z}\right]}{\left(1+\frac{C_s}{365}\right)^{N_s}} \tag{2.22}$$

$$= \max\left[0, \sum_{i=1}^{12}(p_i - x)\right] s_{yz}$$

where

$$s_{yz} = \left[1 - \frac{\left(1+\frac{G_z}{365}\right)^{N_z}}{\left(1+\frac{C_s}{365}\right)^{N_s}}\right]$$

Will the corporation ever prefer to have paid more than its tax liability for the year? An alternative way to ask this question is, does a set of conditions exist under which a corporation would deliberately pay more than the excess of its tax liability for the year over total instalment payments at the remainder due date? For this to occur, s_{yz} must take a negative value, *i.e.*,

$$\left(1+\frac{G_z}{365}\right)^{N_z} > \left(1+\frac{C_s}{365}\right)^{N_s} \tag{2.23}$$

As N_z is strictly less than N_s, a corporation will only deliberately overpay where the prescribed rate, G, is greater than the corporation's cost of capital, C, by an amount large enough to overcome the effect of receiving zero interest for a period. As this could only occur in extreme circumstances (*i.e.*, borrowing at the risk free rate, and not receiving a refund for 2 years), it is assumed in this thesis that it does not occur.

The loss set out in equation (2.22), the loss from waiting for a refund, does not consider the possibility that the corporation could access its refund through a request to Revenue Canada for the refund to be transferred to another account such as the employer source withholding account. Such transfers are purely administrative practice rather than law, and in the past Revenue Canada has discouraged such transfers (Scheuermann [1988]). However, Revenue Canada Information Circular 81-11R3, which was released March 26, 1993, indicates that most restrictions on transfers between accounts will be ended. If Revenue Canada follows these practices, the opportunity loss from waiting for a refund could be almost completely eliminated -- if a corporation decided on the remainder due date that a refund was owing, it could apply to have that amount treated as a payment in respect of employee source withholdings and consequently reduce the payment that it would otherwise pay to that account. As amounts paid under this account are normally significantly larger than the corporate tax liability, the opportunity loss associated with the stub loss could in most cases be eliminated.

2.2.5 The Loss from Instalments

Recall that from equation (2.1) the corporation's loss for any time path of payments (*i.e.*, for any payment series $p_1, p_2, \ldots p_{12}$) is:

$$l(p;x) = U + O + Pen + Stub, \tag{2.24}$$

Substituting equations (2.12), (2.15), (2.18) and (2.22) into equation (2.24), this may be rewritten as,

$$\begin{aligned} l(p;x) = {} & \max\left[0, \sum_{i=1}^{12} (q_i - p_i) g_i\right] \\ & + \sum_{i=1}^{12} (p_i - q_i) c_i \\ & + .50 \cdot \max\left\{0, \sum_{i=1}^{12} (q_i - p_i) g_i - \max\left(1000, .25 \sum_{i=1}^{12} q_i g_i\right)\right\} \\ & + \max\left[0, \sum_{i=1}^{12} \left(p_i - x\right)\right] s_{yz} \end{aligned} \tag{2.25}$$

2.3. An Alternative Formulation

In this thesis, the loss as presented in equation (2.25) (or an expected loss based on that formulation) is minimized in determining the corporation's optimal payment strategy. As stated in the introduction to this chapter, an alternative objective function, which would give the same optimal payment path, would be to minimize the present value of all payments required by the Income Tax Act to be paid by the taxpayer to Revenue Canada in respect of that year (where refunds from Revenue Canada are included as negative payments). Note that this alternative objective function may be equivalently stated as maximizing the present value of the cash flows.

To demonstrate this equivalence, note that the present value of all tax payments, denoted $t_{pv}(p;x)$, may be written,

$$
\begin{aligned}
t_{PV}(p;x) = & \sum_{i=1}^{12} \frac{p_i}{\prod_{j=0}^{i-1}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} \\
& + \frac{\max\left[0, \sum_{i=1}^{12} (q_i - p_i) g_i\right]}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} \\
& + \frac{.50 \cdot \max\left[0, \sum_{i=1}^{12} (q_i - p_i) g_i - \max\left(1000, .25 \sum_{i=1}^{12} q_i g_i\right)\right]}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} \qquad (2.26) \\
& + \frac{\max\left[0, \sum_{i=1}^{12} (x - p_i)\right]}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} \\
& - \frac{\max\left[0, \sum_{i=1}^{12} (p_i - x)\right]}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}} \cdot \left(1+\frac{C_z}{365}\right)^{N_z}} - \frac{\max\left[0, \sum_{i=1}^{12} (p_i - x)\right] \cdot \left[\left(1+\frac{G_z}{365}\right)^{N_z} - 1\right]}{\left[\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}\right] \cdot \left(1+\frac{C_z}{365}\right)^{N_z}}
\end{aligned}
$$

Note that the denominator in each term discounts an amount to the first day of the corporation's fiscal year. The terms have the following interpretation. The first term is the present value of the 12 instalment payments. The numerator in terms two and three are the amounts *U* and *Pen*, the interest liability and penalty loss respectively at the remainder due date. The denominator in these two terms discounts *U* and *Pen* from the remainder due date to the first day of the corporation's tax year. The fourth term represents the present value of the amount of tax owing at the remainder due date (if any), which is given by equation (2.19) above. Terms five and six represent the effects of having paid an amount greater than the tax liability for the year in the instalment period or on the remainder due date; the fifth term is the associated refund and the sixth term is the interest from the government, if any, associated with the overpayment. Both of these amounts are calculated at the refund date and discounted to the first day of the corporation's fiscal year.

In Appendix C, it is demonstrated that minimizing $t_{pv}(p;x)$ will produce the same optimal values of the decision variables, $p_1, p_2, \ldots, p_{12}$ as minimizing $t(p;x)$.

CHAPTER 3

A SINGLE INSTALMENT MODEL

To gain certain insights into the optimal behaviour of corporations, a single instalment model is developed. The loss from instalments in this model is developed in section 3.1 as a special case of the loss set out in equation (2.25) of chapter 2. Section 3.2 determines the optimal instalment.

3.1 Loss from a Single Instalment Model

Consider a model in which the corporation has an instalment liability at only one date in the year (instead of at 12 dates in the year), and makes a payment at only that date. Denoting this payment date as j, the restriction on instalment liability and payment dates are;

$$\begin{aligned} q_i &= 0 \quad \forall\, i \neq j \\ p_i &= 0 \quad \forall\, i \neq j \end{aligned} \tag{3.1}$$

Assume also that the instalment liability at date j is the lesser of the tax liability for the year and the first instalment base; *i.e.*,

$$q_j = \min\left(12x, 12b_1\right) \tag{3.2}$$

since x and b_1 are defined in the preceding chapter as 1/12 of the corporation's tax liability for the year and first instalment base respectively. For convenience of notation, let $X = 12x$ be the tax liability for the year, and let $B_1 = 12b_1$ be the first instalment base. Then the corporation's instalment liability may be written;

$$q_j = \min(X, B_1) \tag{3.3}$$

A simplification used in obtaining equations (3.2) and (3.3) is that paying based on the second instalment base is either not an alternative in law, or is irrelevant as the first instalment base is less than the second instalment base.[26] Assume further that any amount overpaid in the instalment period is refunded by Revenue Canada at the remainder due date. Thus, there is no stub loss.

With the above assumptions, the loss from chapter 2 (equation (2.25)), for any payment p, may be rewritten,

$$\begin{aligned} l(p_j;X) = {} & \max\left[0,\ (q_j - p_j)g_j\right] \\ & + (p_j - q_j)c_j \\ & + .50 \cdot \max\left[0,\ (q_j - p_j)g_j - \max(1000, .25\, q_j g_j)\right] \end{aligned} \tag{3.4}$$

For convenience, the subscript j is deleted for the remainder of this chapter, *i.e.*,

[26]This single instalment structure is directly relevant to a class of Canadian individuals who make only a single instalment payment, farmers and fishermen. By section 155 of the federal Income Tax Act, a farmer or fisherman who is subject to the instalment rules must pay by December 31 of each year the lesser of 2/3 of the individual's tax liability for the current year, and 2/3 of the individual's tax liability for the preceding year. The balance is then payable on or April 30 of the following year, which is the remainder due date.

$$l(p;X) = \max[0, (q-p)g] + (p-q)c + .50 \cdot \max[0, (q-p)g - \max(1000, .25qg)] \tag{3.5}$$

As,

$$(p-q)c = \max[0,(p-q)c] - \max[0,(q-p)c], \tag{3.6}$$

equation (3.5) may be rewritten,[27]

$$l(p;X) = \max[0, q-p](g-c) + \max[0, p-q]\, c + .50 \cdot \max[0, q-p)g - \max(1000, .25 \min(X, B_1)\, g)]. \tag{3.7}$$

This equivalent expression, which is used throughout the remainder of this chapter, has a nice interpretation. The first term is the loss from underpayment (excluding the penalty), and the second term is the loss from overpayment. Note the symmetry between the definitions of overpayment and underpayment in equation (3.7): if one is positive, the other must be zero. An overpayment and underpayment cannot occur together. The third term remains the penalty loss.

The loss to the corporation from an underpayment is the amount of the underpayment multiplied by the cost to the corporation of a dollar of underpayment. This is simply,

$$(q-p) \cdot (g-c) \tag{3.8}$$

where q-p is the amount of underpayment, and the loss from a dollar of underpayment is the

[27]In the multi-period setting, the first two terms in the loss function, U and O, cannot be rewritten as the sum of two mutually exclusive amounts.

difference between the after-tax rate of interest that the corporation pays to the government on underpayments, g, and the corporation's after-tax cost of capital, c.

The loss to the corporation from an overpayment is the amount of the overpayment multiplied by the cost to the corporation of a dollar of overpayment. This is simply,

$$(p-q)\cdot c \tag{3.9}$$

where p-q is the amount of overpayment, and the loss from a dollar of overpayment is the corporation's after-tax cost of capital, c.

3.2 The Optimal Payment

If the corporation knew with certainty its tax liability for the year, it could minimize its loss through paying an amount equal to its instalment liability, $p = q$, given $g > c > 0$. This is an optimal solution[28] as,

a. if $p = q$, then $l(p;X) = 0$,[29] and

b. $l(p;X) \geq 0$.[30]

---

[28]This result would not change with the inclusion of the stub loss into the one period model. However, this result is not true in a multiperiod model with compound interest (see section 4.4).

[29]From equation (7), it is easy to see that where $q = p$, each of the three terms equals zero.

[30]The loss will be bounded at zero if each of the three terms in equation (7) is non-negative. The first term is non-negative as $g - c \geq 0$. The second term is non-negative as $c \geq 0$. The final term is non-negative by construction.

Therefore, if the corporation knows its tax liability for the current year, it should pay the lesser of that amount and its preceding year's tax liability. However, where the corporation's tax liability for the current year is uncertain, such simple prescriptions for behaviour do not exist.

Let us assume that the corporation is risk-neutral. The corporation's objective is therefore to choose a payment amount p so as to minimize the expected loss function $L(p)$:

$$L(p) = \int_0^{+\infty} l(p;X) f(X)\, dX. \tag{3.10}$$

In the above definition, $f(X)$ is the density function for the tax liability, and $l(p,X)$ is the loss defined in equation (3.7) above. An assumption made about the nature of this density function, $f(X)$, is that density begins at zero; this assumption is utilized as instalment liability cannot be negative.

A significant weakness of the above formulation of the expected loss function is that it is assumed that the probability of zero tax liability for the year is zero, *i.e.*, $\int_0^0 f(X)\, dX = 0$.

Since all corporations with tax losses for the year will have a zero tax liability, it would be more appropriate to assign a positive probability to a zero tax liability. This more realistic assumption is made in chapter 4.

3.2.1 No Deliberate Overpayment

Although the tax liability, X, is unknown at the time the instalment payment, p, is made, it is possible to place an upper bound on the instalment payment. As instalment liability is defined as $\min(X,B_1)$, the corporation's instalment liability must be less than or equal to its tax liability for the preceding year, B_1. Therefore, any instalment payment in excess of B_1 will, with certainty, create an additional loss to the corporation in relation to the amount of the payment in excess of B_1. In effect, a corporation which pays in excess of B_1 is "burning money". The Lemma below demonstrates that if $p > B_1$, a $1 decrease in the instalment payment decreases the corporation's expected loss by the unit cost of overpayment, c.

Lemma: If $p > B_1$, then $dL/dp = c > 0$.

Proof:

From equation (3.7), there is no penalty if $p > B_1$. The logic is as follows:

$$
\begin{aligned}
p > B_1 \Rightarrow \; & p > \min(X,B_1) \\
\Rightarrow \; & \min(X,B_1) - p < 0 \\
\Rightarrow \; & (\min(X,B_1) - p)g - \max(1000, .25 \min(X,B_1)\, PI) < 0 \\
\Rightarrow \; & Pen = 0
\end{aligned}
$$

From equation (3.7), $p > B_1$ also implies that the underpayment (and thus the loss from underpayment) is also zero:

$$
\begin{aligned}
p > B_1 \Rightarrow\ & p > \min(X,B_1) \\
\Rightarrow\ & \min(X,B_1) - p < 0 \\
\Rightarrow\ & (g-c)\cdot\max[0, (\min(X,B_1)-p)] = 0 .
\end{aligned}
$$

Therefore, from (3.7), the loss reduces to the loss from overpayment

$$l(p;X) = c\cdot\max(0, p-\min(X, B_1))$$

which can be rewritten as follows:[31]

$$l(p;X) = \begin{cases} c\,(p-B_1) & \text{if } X \ge p \ge B_1 \\ c\,(p-B_1) & \text{if } p \ge X \ge B_1 \\ c\,(p-X) & \text{if } p \ge B_1 \ge X \end{cases} \tag{3.11}$$

The three branches of this function are created by the fact that X, given $p > B_1$, can have three possible values relative to p and B_1: X may be greater than or equal to p, X may be between p and B_1 (*i.e.*, $p \geq X \geq B_1$), or X may be less than or equal to B_1.

Substituting equation (3.11) into equation (3.10), the expected loss function for $p > B_1$ is:

[31]To demonstrate the method used in the derivation, consider the top branch of the function: since $X \geq p \geq B_1$, $\min(X,B_1) = B_1$, and $p - \min(X,B_1) = p - B_1$. Therefore, $l(p;X) = c(p-B_1)$. The other two branches may be derived similarly.

$$L(p\,|\,p>B_1) = c\int_{p}^{+\infty}(p-B_1)f(X)\,dX + c\int_{B_1}^{p}(p-B_1)f(X)\,dX + c\int_{0}^{B_1}(p-X)f(X)\,dX\,. \tag{3.12}$$

Differentiating[32] equation (3.12) with respect to p[33];

[32]To differentiate this function with respect to p, it is convenient to use the following rule for differentiation of an integral with respect to a parameter:

$$\frac{d}{da}\int_{p}^{q} g(x,a)\,dx = \int_{p}^{q}\frac{\partial}{\partial a}[g(x,a)]\,dx + g(q,a)\frac{dq}{da} - g(p,a)\frac{dp}{da}\,.$$

See Apostol [1957,219].

[33]This result is determined as follows:

$$\begin{aligned}\frac{dL(p\,|\,p>B_1)}{dp} &= c\,[-(p-B_1)f(p) + \int_{p}^{+\infty} f(X)\,dX] \\ &\quad + c\,[(p-B_1)f(p) + \int_{B_1}^{p} f(X)\,dX] \\ &\quad + c\int_{0}^{B_1} f(X)\,dX \\ &= c\int_{B_1}^{+\infty} f(X)\,dX + c\int_{0}^{B_1} f(X)\,dX \\ &= c\int_{0}^{+\infty} f(X)\,dX \\ &= c\,.\end{aligned}$$

$$\frac{dL(p \mid p>B_1)}{dp} = c.$$

This derivative is strictly positive. Also, $L(p)$ is continuous at $p=B_1$ because the limits from the left and the right are equal.[34] Therefore, it is never optimal to deliberately pay more than B_1, and hence, it is possible to eliminate the region where $p>B_1$.[35] Minimizing the expected loss function $L(p)$ subject to $p \geq 0$ is therefore equivalent to minimizing the expected loss function subject to $0 \leq p \leq B_1$. Therefore, throughout the remainder of this paper p is restricted to be less than or equal to B_1.

The intuitive explanation of this result is that any instalment payment amount greater than B_1 must include some overpayment. For a dollar paid greater than B_1, the corporation loses with certainty the unit cost of overpayment, c. Therefore, it is never optimal to deliberately pay more than B_1. Hence, the region where $p>B_1$ can be eliminated.

[34]The limits, both from the left and from the right, equal

$$c\int_0^{B_1}(B_1-X)f(X)\,dX.$$

[35]Because of the strictly positive derivative and continuity, for any p greater than B_1 there exists a feasible perturbation (*i.e.* to decrease p) which will decrease the objective function. Hence, $p>B_1$ does not satisfy the Diewert [1981] necessary conditions for an optimum. See Macnaughton [1993] for details. These conditions will be examined in some detail in Appendix B.

3.2.2 The Pre-1990 Problem

Prior to 1990, there was no penalty in Canada for underpayment of instalments: a corporation which underpaid was only liable for interest at the prescribed rate. Although the penalty forms part of the current law, it is worthwhile to examine the problem without penalty for the following reasons: first, additional intuition may be gained through developing that simplified model; second, examining the model without penalty allows for comparative study (to examine the impact of the introduction of the penalty provision); and third, the simplified model reflects the United States instalment (estimated tax) system in which there is no penalty.

Rewriting equation (3.7) without the penalty, the loss is,

$$l(p;X) = (g-c) \cdot \max(0, \min(X, B_1)-p) + c \cdot \max(0, p-\min(X, B_1)).$$

As in the above, it is useful to rewrite the loss, given $p \leq B_1$, as follows:

$$l(p;X) = \begin{cases} (g-c) \cdot (B_1-p) & \text{for } X \geq B_1 \geq p \\ (g-c) \cdot (X-p) & \text{for } B_1 \geq X \geq p \\ c \cdot (p-X) & \text{for } B_1 \geq p \geq X \end{cases} \tag{3.14}$$

Diagrammatically, equation (3.14) may be presented as in Figure 3.1.

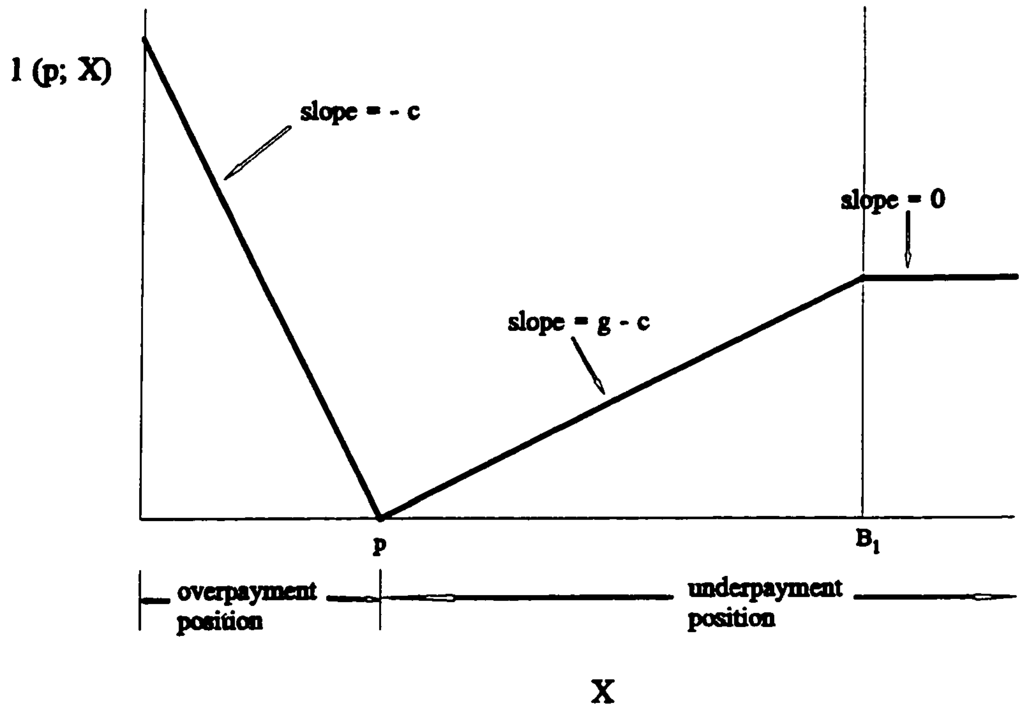

FIGURE 3.1
Pre-1990 Loss

In this figure the instalment payment, p, is fixed and the effect of different levels of tax liability, X, on loss is examined. Two points may be made with respect to the structure of the loss in Figure 3.1. First, a fundamental asymmetry between overpayment and underpayment may be recognized through examining the slopes on underpayment and overpayment: the loss from $1 of overpayment, c, is not generally equal to the loss from $1 of underpayment, g-c. Second, the loss from underpayment for any value of X greater than B_1 is a constant amount

$(g\text{-}c)(B_1\text{-}p)$.

Recall that the optimization problem is:

$$\min_p L(p) \quad \textit{subject to} \quad 0 \le p \le B_1 . \tag{3.15}$$

From equation (3.14), the expected loss function for this range of p is

$$\begin{aligned} L(p) &= (g-c) \int_{B_1}^{+\infty} (B_1 - p) f(X)\, dX \\ &+ (g-c) \int_{p}^{B_1} (X-p) f(X)\, dX \\ &+ c \int_{0}^{p} (p-X) f(X)\, dX \end{aligned} \tag{3.16}$$

The derivative of this function[36] (again using the rule for differentiation of an integral

[36]The function $L(p)$ is continuously differentiable as it is the sum of integrals, each continuously differentiable, and the sum of continuously differentiable functions is itself continuously differentiable (Adams [1990,63]).

with respect to a parameter) with respect to p is;

$$\frac{dL(p)}{dp} = -(g-c)\int_{B_1}^{+\infty} f(X)\,dX$$

$$-(g-c)\int_{p}^{B_1} f(X)\,dX$$

$$+\,c\int_{0}^{p} f(X)\,dX$$

$$= c\int_{0}^{p} f(X)\,dX \; - \; (g-c)\int_{p}^{+\infty} f(X)\,dX \qquad (3.18)$$

Intuition may be provided for this derivative. Consider an increase of $1 in the instalment payment. If the corporation is in an overpayment position, this increases the loss by the unit cost of overpayment, c. If the corporation is in an underpayment position, this decreases the loss by the unit cost of underpayment, g-c. These losses must be weighted by the probabilities that they will occur; where $\int_0^p f(X)\,dX$ is the probability of overpayment, and $\int_p^{+\infty} f(X)\,dX$ is the probability of underpayment. Thus, equation (3.18) states that the effect of an increase of $1 in the instalment payment is the difference between (1). the unit cost of overpayment, multiplied by the probability of overpayment and (2). the unit cost of underpayment, multiplied by the probability of underpayment.

From equation (3.18), the second derivative of the expected loss function is:

$$\frac{d^2 L(p)}{dp^2} = g\, f(p) > 0. \tag{3.19}$$

Therefore the expected loss function $L(p)$ is convex. Thus the optimization problem as set out in (3.15) above is one of minimizing a convex function subject to linear constraints. Hence, the Kuhn-Tucker conditions for that problem are both necessary and sufficient for a global minimum[37].

The Kuhn-Tucker conditions may be written as follows:

$$p^* = 0 \qquad if \quad \frac{dL(0)}{dp} \geq 0 \tag{3.20}$$

$$0 < p^* < B_1 \qquad if \quad \frac{dL(p^*)}{dp} = 0 \tag{3.21}$$

$$p^* = B_1 \qquad if \quad \frac{dL(B_1)}{dp} \leq 0 \tag{3.22}$$

[37]See Avriel et al. [1988].

From (3.18) the condition from (3.20) simplifies to[38]:

$$p^* = 0 \qquad if \quad g \leq c \tag{3.23}$$

This condition is not feasible as it is assumed throughout this thesis that $g > c$; the interpretation of this condition is that one would deliberately underpay (deliberately borrow from the government through not paying any instalment) where the cost from underpayment was less than the corporation's after-tax cost of capital. The effect of the assumption that $g > c$ is therefore that the corporation will make a strictly positive instalment payment.

From (3.18) the conditions from (3.21), and (3.22) simplify to:

$$0 < p^* < B_1 \qquad if \quad (g-c)\int_{p^*}^{+\infty} f(X)\,dX = c\int_{0}^{p^*} f(X)\,dX \tag{3.24}$$

$$p^* = B_1 \qquad if \quad (g-c)\int_{B_1}^{+\infty} f(X)\,dX \geq c\int_{0}^{B_1} f(X)\,dX \tag{3.25}$$

Let us interpret these feasible conditions. The second condition, equation (3.25), states that paying less than B_1 is not optimal if this will increase the expected loss from underpayment by as much or more than it decreases the expected loss from overpayment. Hence B_1 is the optimum in this situation. The first condition, (3.24), states that if for some instalment payment p no benefit can be obtained by either increasing or decreasing the payment, then that

[38]To obtain (3.23), note that,

$$\frac{dL(0)}{dp} = -(g-c)\int_{0}^{+\infty} f(X)\,dX = -(g-c).$$

amount of instalment payment is the optimum.

It may be noted that the interior solution above (equation (3.24)) may be rewritten such that it is mathematically equivalent, within a specific range, to the news vendor solution from the management sciences literature:[39]

$$\int_0^{p^*} f(X)\,dX = \frac{g-c}{g} \qquad \text{if } 0 < p^* < B_1 . \tag{3.26}$$

The optimal probability of overpayment for an interior solution (where the boundaries are not binding) is equivalent to the critical fractile (the optimal probability of "stocking out") derived in the newsvendor literature. In that literature the critical fractile is the ratio of the unit underage cost from having too little inventory (in our model underpayment) to the sum of the unit underage and unit overage costs (in our model the unit cost from underpayment, g-c, plus the unit cost from overpayment, c). Hence, the pre-1990 problem would reduce to the news vendor problem if the other feasible solution could be ruled out: that is, if $E_1 = +\infty$. The news vendor problem is therefore a special case of the simplified single instalment pre-1990 problem considered here.

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[39]For a summary of this literature, see Porteus [1990].

3.2.3. The Post-1989 Problem

For tax years 1990 and after, the penalty under section 163.1 of the Income Tax Act applies. Recall that the loss with this penalty is, from equation (3.7):

$$\begin{aligned} l(p;X) &= (g-c)\max(0, \min(X, B_1)-p) \\ &+ c\max(0, p-\min(X, B_1)) \\ &+ .50\max[0, (\min(X,B_1)-p)g - \max(1000, .25\min(X,B_1)\,g)] \end{aligned} \tag{3.27}$$

To solve for an optimal p, it is necessary to eliminate the maximization and minimization operators, and to partition regions in p over which different consequences occur. These partitions are structured such that any feasible p (any value of p such that $0 \le p \le B_1$) will fall in a single region, and for a value of p in that region, the loss may be determined for all values of X. This result is derived in Appendix D.

The loss is partitioned as follows,

$$l(p;X) = \begin{cases} e_1 & \text{if } p \le \frac{3000}{g} \text{ and } B_1 \ge \frac{4000}{g} \\ e_2 & \text{if } p \le \frac{3000}{g} \text{ and } \frac{4000}{g} \ge B_1 \ge p + \frac{1000}{g} \\ e_3 & \text{if } p \le \frac{3000}{g} \text{ and } p + \frac{1000}{g} \ge B_1 \\ e_4 & \text{if } p \ge \frac{3000}{g} \text{ and } B_1 \ge \frac{p}{.75} \\ e_5 & \text{if } p \ge \frac{3000}{g} \text{ and } B_1 \le \frac{p}{.75} \end{cases} \tag{3.28}$$

where e_1 to e_5 are defined below. The loss for each region (*i.e.*, each pair of inequalities) is examined separately. To gain insight, an explanation for the form of the loss in each loss region will be provided with accompanying diagrams.

To facilitate understanding, it is useful to consider the penalty as consisting of two separate calculations. The first calculation, (*i*), applies where subsection 163.1(b) of the Act is applicable: the penalty is 50% of the amount, if any, by which interest payable in respect of instalments exceeds $1,000. The second calculation, (*ii*), applies where subsection 163.1(c) of the Act is applicable: the penalty is 50% of the amount, if any, by which interest payable in respect of instalments exceeds 25% of the instalment interest that would have been payable for the year if no instalment payment had been made for the year. Algebraically, these penalty calculations, (*i*) and (*ii*), are:

$$(i) \quad .50\,[(\min(X, B_1) - p)g - 1000]\ ;$$

and

$$(ii) \quad .50\,[(\min(X, B_1) - p)g - .25\min(X, B_1)g], \qquad or$$
$$.375\,(\min(X, B_1) - p)\,g - .125\,p\,g$$

respectively.

Where $p \leq 3000/g$, either penalty calculation could be operative, depending on the value of p. Decreasing p from B_1 to zero, the first calculation, (*i*), is always operative prior to the second calculation, (*ii*), becoming operative. Where $p \geq 3000/g$, only the second calculation, (*ii*), can become operative. The second set of constraints in (3.28) is required as the expected loss function changes across regions defined by the relationship between p and B_1.

For expositional purposes, we will examine the loss region e_4 prior to examining e_1, e_2, e_3 and e_5. Loss region e_4 occurs where the corporation makes an instalment payment p

which is greater than $3000/g$, and the inequality $B_1 \geq p/.75$ holds. In that region penalty calculation (*i*) can never be in operative. Penalty calculation (*ii*) is operative where the corporation underpays by more than 25%. Where the penalty is operative, the corporation pays a 37.5% penalty on interest from underpayment. An implication is that corporations with instalment liability greater than $3000/g$, can avoid penalties (can have zero probability of penalty), if they pay $.75B_1$.

The loss for the fourth region, e_4, is

$$e_4 = \begin{cases} (1.5g-c)(B_1-p) - .125gB_1 & \text{if} \quad X \geq B_1 \\ (1.5g-c)(X-p) - .125gX & \text{if} \quad B_1 \geq X \geq \frac{p}{.75} \\ (g-c)(X-p) & \text{if} \quad \frac{p}{.75} \geq X \geq p \\ c(p-X) & \text{if} \quad p \geq X \end{cases} \tag{3.29}$$

This loss is presented diagrammatically in Figure 3.2. Note that in the diagrams in this section, the instalment payment, p, is fixed and the effect of different levels of tax liability, X, on loss is examined.

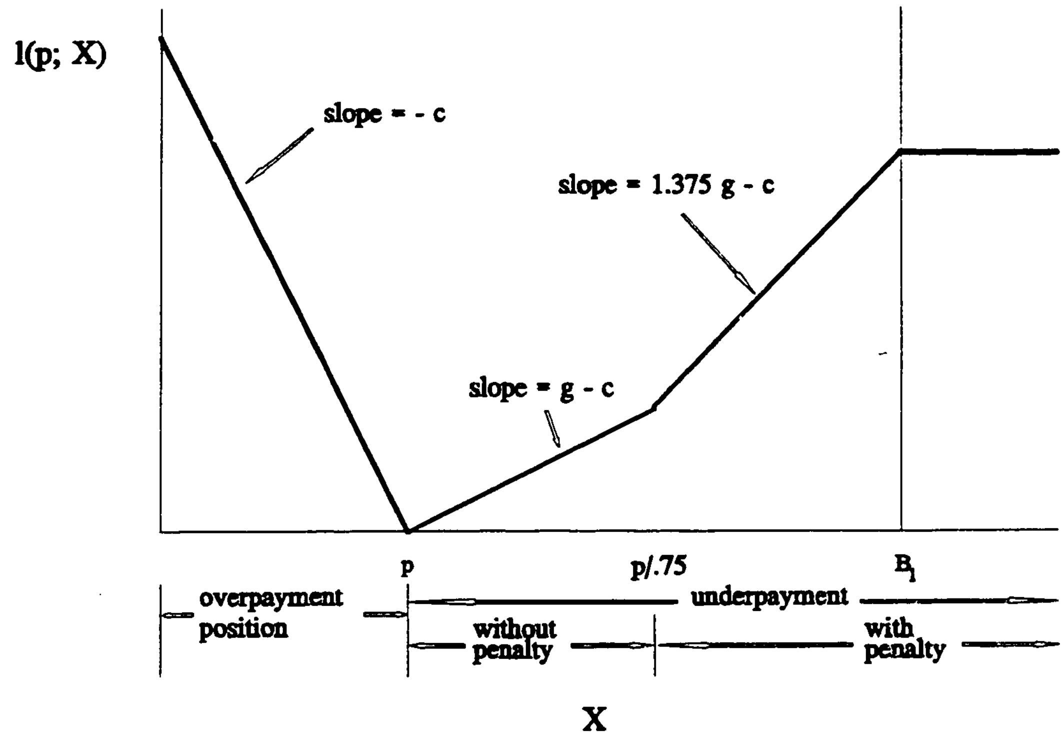


FIGURE 3.2
The Loss for the Fourth Region, e_4

Let us now examine the first loss region, e_1. In this region, $p \leq 3000/g$ and $B_1 \geq 4000/g$. The loss is:

$$e_1 = \begin{cases} (1.5g-c)(B_1-p) - .125gB_1 & \text{if } X \geq B_1 \\ (1.5g-c)(X-p) - .125gX & \text{if } B_1 \geq X \geq \frac{4000}{g} \\ (1.5g-c)(X-p) - 500 & \text{if } \frac{4000}{g} \geq X \geq p+\frac{1000}{g} \\ (g-c)(X-p) & \text{if } p+\frac{1000}{g} \geq X \geq p \\ c(p-X) & \text{if } p \geq X \end{cases} \tag{3.30}$$

This is represented diagrammatically in Figure 3.3. Note that the penalty does not become active until the instalment liability X is greater than the amount paid by $1000/$g$. If the corporation underpays such that the interest on underpayment is greater than $1000 ($X > p + 1000/g$), it will pay a penalty. The penalty rate at that point is 50% of each additional dollar of interest. However, if the corporation underpays by a large amount (by at least 25%) the penalty declines to 37.5% of each additional dollar of interest. As in the no penalty case, the loss reaches a maximum where X equals B_1.

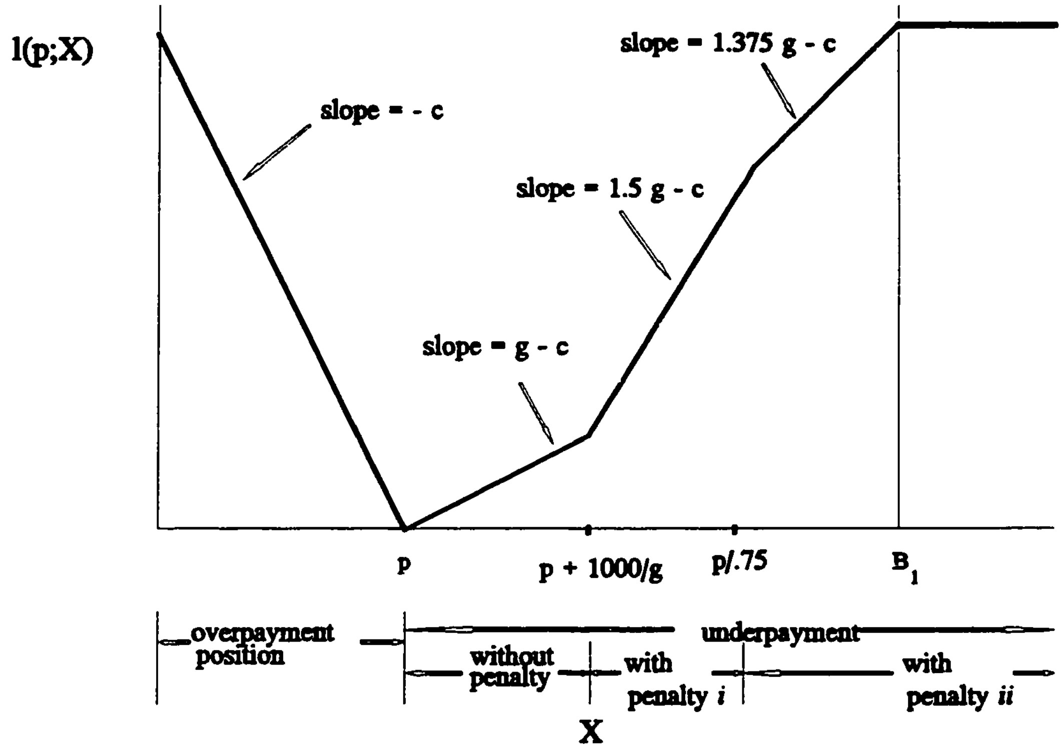


FIGURE 3.3

The Loss for the First Region, e_1

Let us now examine the second loss region, e_2:

$$e_2 = \begin{cases} (1.5g-c)(B_1-p)-500 & \text{if } X \geq B_1 \\ (1.5g-c)(X-p)-500 & \text{if } B_1 \geq X \geq p+\dfrac{1000}{g} \\ (g-c)(X-p) & \text{if } p+\dfrac{1000}{g} \geq X \geq p \\ c(p-X) & \text{if } p \geq X \end{cases} \tag{3.31}$$

This region, as presented in Figure 3.4, is defined by $p \leq 3000/g$ and $4000/g \geq B_1 \geq$

$p+1000/g$. There is a potential to underpay such that penalty calculation (*i*) can apply ($p+1000/g - B_1 \leq 0$), but penalty calculation (*ii*) can never apply given the restriction on B_1 ($p/.75 - B_1 > 0$). The penalty, where operative, is at a 50% rate.

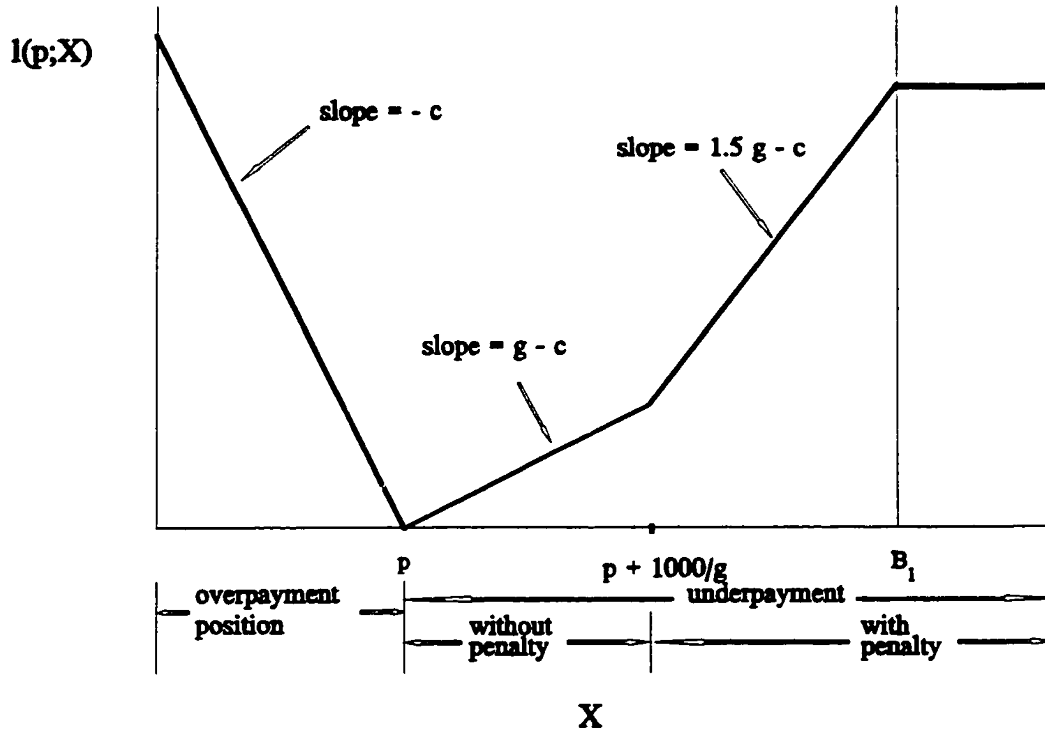


FIGURE 3.4

The Loss for the Second Region, e_2

Note that we have not yet examined the loss for e_3 and e_5. The loss for each may be written:

$$e_3 = e_5 = \begin{cases} (g-c)(B_1-p) & \text{if} \quad X \geq B_1 \\ (g-c)(X-p) & \text{if} \quad B_1 \geq X \geq p \\ c(p-X) & \text{if} \quad p \geq X \end{cases} \tag{3.32}$$

For each of these ranges the probability of incurring penalties, for an instalment payment p, is zero. This occurs as B_1 is sufficiently low in relation to p. The probability of incurring penalties is zero where the corporation pays an amount less than $3000/g$ and greater than B_1-$1000/g$, or where the corporation pays an amount greater than $3000/g$ and greater than $B_1/.75$. Figure 1 (the pre-1990 loss) is therefore applicable to these ranges.

Solution

Recall that the expected loss function is defined as:

$$L(p) = \int_0^{+\infty} l(p;X)f(X)\,dX. \tag{3.33}$$

The problem is to minimize this function through the choice of an instalment payment amount in the range from zero to the barrier amount B_1. Because $L(p)$ is not continuously differentiable, this minimization problem cannot be solved by traditional methods of calculus. Appendix F solves these problems using a method developed by Diewert [1981] and Macnaughton [1993] for optimization problems in which the functions involved have kinks. This method is based on the principle that at the optimum it must not be possible to find a feasible perturbation of the value of the decision variable which decreases the value of the objective function. This generalizes the usual calculus conditions to cover optimal points which may be interior solutions but which also may be at the boundary of the feasible region or at kinks.

The conditions for an optimum differ according to the range of parameter values being considered. For $B_1 \leq 4000/g$[40], the conditions for an optimum are:

$$(i)\quad p^* = 0 \qquad if \quad c \geq g + .5g \int_{1000/g}^{\infty} f(X)\,dX$$

$$(ii)\quad 0 < p^* < B_1 - \frac{1000}{g} \qquad if \quad .5g \int_{p^*+1000/g}^{\infty} f(X)\,dX + (g-c)\int_{p^*}^{\infty} f(X)\,dX = c\int_0^{p^*} f(X)\,dX$$

$$(iii)\quad p^* = B_1 - \frac{1000}{g} \qquad if \quad -.5g\int_{B_1}^{\infty} f(X)\,dX - (g-c)\int_{B_1-1000/g}^{\infty} f(X)\,dX + c\int_0^{B_1-1000/g} f(X)\,dX \leq 0 \leq -(g-c)\int_{B_1-1000/g}^{\infty} f(X)\,dX + c\int_0^{B_1-1000/g} f(X)\,dX \tag{3.34}$$

$$(iv)\quad p^* > B_1 - \frac{1000}{g} \qquad if \quad (g-c)\int_{p^*}^{\infty} f(X)\,dX = c\int_0^{p^*} f(X)\,dX$$

$$(v)\quad p^* = B_1 \qquad if \quad (g-c)\int_{p^*}^{\infty} f(X)\,dX \geq c\int_0^{p^*} f(X)\,dX$$

[40]For a subset of this set of parameter values, $B_1 \leq 1000/g$, the optimal solution collapses to the pre-1990 (no penalty) solution (equations (3.23), (3.24), and (3.25)). Intuitively, if $B_1 \leq 1000/g$, it is impossible to have a penalty regardless of the value of X. As these solutions are fully derived and discussed earlier in this paper, they are not further examined here.

Similarly, for $B_1 \geq 4000/g$ the conditions for an optimum are:

$$(i)\quad p^* = 0 \qquad if \quad c \geq g + .5g \int_{1000/g}^{\infty} f(X)\,dX$$

$$(ii)\quad 0 < p^* < .75B_1 \qquad if \quad .5g \int_{p^*/.75}^{\infty} f(X)\,dX + (g-c)\int_{p^*}^{\infty} f(X)\,dX = c\int_{0}^{p^*} f(X)\,dX$$

$$(iii)\quad p^* = .75B_1 \qquad if \quad -.5g\int_{B_1}^{\infty} f(X)\,dX - (g-c)\int_{.75B_1}^{\infty} f(X)\,dX + c\int_{0}^{.75B_1} f(X)\,dX \leq 0 \leq -(g-c)\int_{.75B_1}^{\infty} f(X)\,dX + c\int_{0}^{.75B_1} f(X)\,dX \tag{3.35}$$

$$(iv)\quad p^* > .75B_1 \qquad if \quad (g-c)\int_{p^*}^{\infty} f(X)\,dX = c\int_{0}^{p^*} f(X)\,dX$$

$$(v)\quad p^* = B_1 \qquad if \quad (g-c)\int_{p^*}^{\infty} f(X)\,dX \geq c\int_{0}^{p^*} f(X)\,dX$$

The conditions for the two ranges of parameter values may be discussed together. Recall that by assumption $g > c$. Therefore, condition (*i*) in equations (3.34) and (3.35) are not feasible -- the corporation will pay a strictly positive amount.

Let us examine the condition under which the optimal instalment payment is at the boundary of the feasible region; at B_1 (condition (v) in equations (3.34) and (3.35)). The optimal payment, $p^* = B_1$, is identical to the condition under the pre-1990 rules ((3.25) above). Under this condition, paying less than B_1 is not optimal because this would increase the expected loss from underpayment by as much or more than it decreases the expected loss from overpayment. Recall that the strategy to pay more than B_1 was eliminated in section 2.1. The penalty does not enter into these marginal conditions as there is zero probability of penalty at B_1.

Let us now examine the interior solutions, conditions (*ii*) and (*iv*) from equations (3.34) and (3.35). In examining condition (*ii*), the effect of an increase of $1 in the instalment payment p is to increase the expected loss from overpayment by the unit cost of overpayment, c, multiplied by the probability of overpayment. There is an accompanying decrease in the expected loss from underpayment (the unit cost of underpayment multiplied by the probability of underpayment) plus an expected decrease in the penalty (.5 x g x the probability of penalty). The optimal payment, p^* will therefore be chosen such that the marginal expected loss from overpayment just equals the marginal expected loss from underpayment. For the second interior solution, condition (*iv*), it is known with certainty that the penalty cannot apply. The second interior solution is therefore identical to the condition under the pre-1990 rules (equation (3.24) above). The effect of an increase of $1 in the instalment payment p is to increase the expected loss from overpayment by the unit cost of overpayment, c, multiplied by the probability of overpayment, and to decrease the expected loss from underpayment by the unit cost of underpayment, $g-c$, multiplied by the probability of underpayment. These interior conditions therefore state that if for some instalment payment p no decrease in expected loss can be obtained by either increasing or decreasing the payment, then that amount of instalment payment is the optimum.

Let us now examine the kink solutions, condition (*iii*) from equations (3.34) and (3.35). Consider the exact value of p where the penalty goes from having positive probability to zero probability. If the corporation paid \$1 less, the cost benefit comparison would be the same as described above for condition (*ii*): there would be a decrease in the expected loss from overpayment (the unit cost of overpayment multiplied by the probability of overpayment), and an increase in the expected loss from (1). underpayment (the unit cost of underpayment multiplied by the probability of underpayment) and (2). penalty from underpayment (.5 multiplied by the interest rate on underpayment multiplied by the probability of penalty). If the corporation paid \$1 more, the cost benefit comparison would be the same as described above for condition (*iv*): there would be an increase in the expected loss from overpayment (the unit cost of overpayment multiplied by the probability of overpayment); and a decrease in the expected loss from underpayment (the unit cost of underpayment multiplied by the probability of underpayment).

The kink therefore exists at the value of p which separates a zero probability of penalty (values of p relative to B_1 such that it is impossible for a penalty to occur) from a positive probability of penalty[41]: at $p = B_1 - 1000/g$ given $B_1 \leq 4000/g$; and at $p = .75B_1$ given $B_1 \geq 4000/g$. For example, where $B_1 \geq 4000/g$, there is zero probability of penalty if the

[41]In section 3.2.1 it was demonstrated that there should be no deliberate overpayment - *i.e.*, that $p \leq B_1$. It should be noted that there existed a kink at $p = B_1$ (moving from a positive, but less than 1, probability of overpayment, to a probability of 1 on overpayment). However, as values of $p > B_1$ were strictly dominated by $p = B_1$, that kink could be ignored.

corporation pays an amount greater than three-quarters of the instalment liability for the preceding year ($p > .75B_1$). However, if the corporation pays one dollar less than three-quarters of the instalment liability for the preceding year, there is a positive probability that the corporation will have to pay a penalty on that dollar. Note that these kinks occur as a result of the barrier B_1: if there was not a barrier, the expected loss function would be differentiable for all positive values of p.

To facilitate understanding, the five conditions in equations (3.34) and (3.35), including the first condition which is infeasible by assumption, are presented diagrammatically in Figure 3.5. As discussed above, (*v*) represents the conditions for optimal solutions at the boundary of the feasible region, (*ii*) and (*iv*) represent the conditions for optimal interior solutions, and (*iii*) represents an optimal solution at the kink.

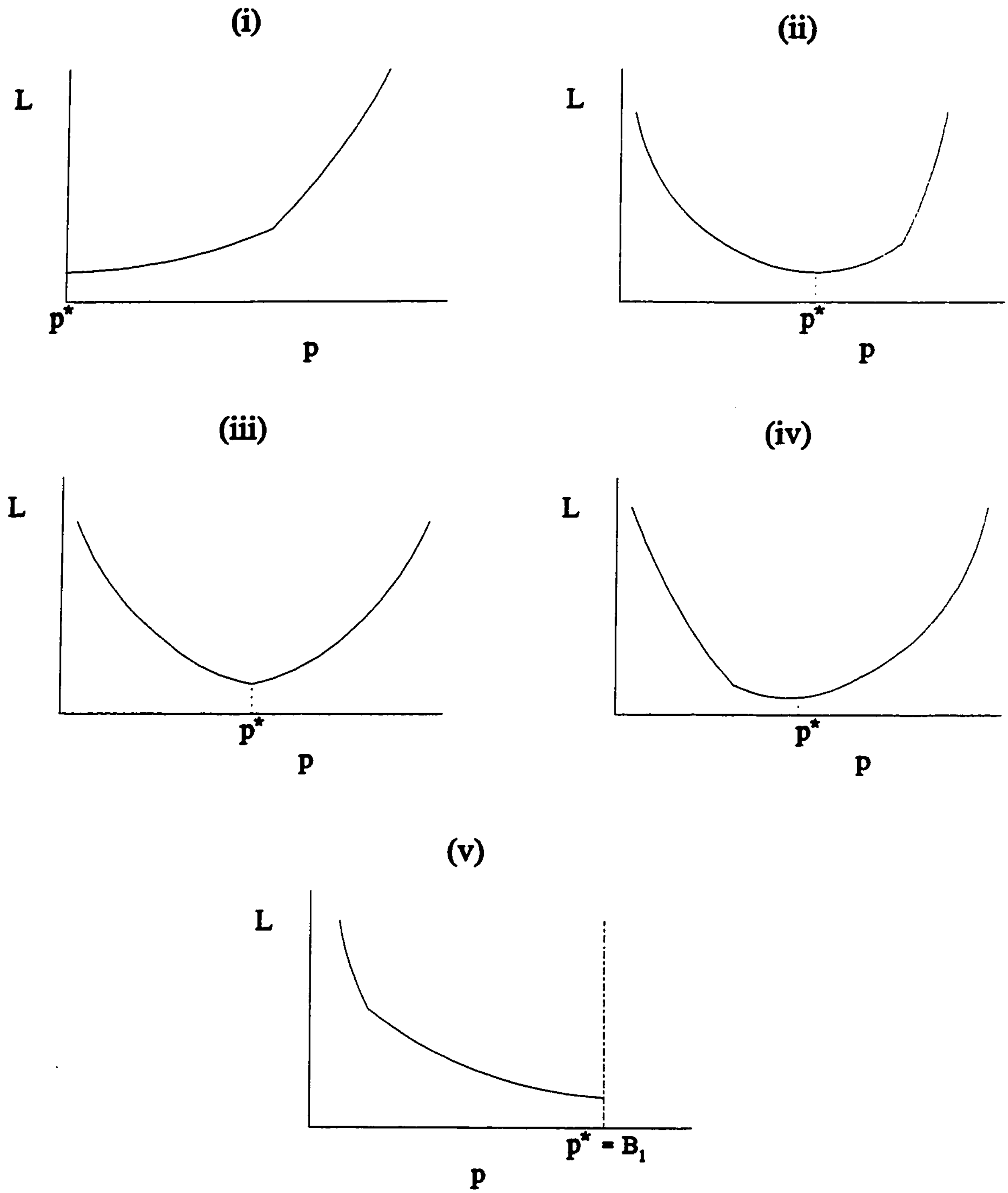


FIGURE 3.5

Five Conditions for an Optimum

CHAPTER 4

MULTI-PERIOD ANALYTICS

This chapter is structured as follows. Section 4.1 sets out the corporation's objective function. In section 4.2, the derivatives of this function are calculated. The conditions for an optimum are then developed in section 4.3. To isolate specific effects, sections 4.4 through 4.6 determine optimal payment amounts for certain restricted problems: section 4.4 focuses on an information effect; section 4.5 examines the effect of the stub loss; and section 4.6 examines the effect of compound rates.

4.1 The Corporation's Objective Function

Recall from chapter two that the corporation's loss for any monthly series of instalment payments, $p_1, p_2, \ldots, p_{12}$, and any monthly-average tax liability for the year, x, is,

$$
\begin{aligned}
l(p;x) = {} & \max\left[0, \sum_{i=1}^{12} (q_i - p_i) g_i\right] \\
& + \sum_{i=1}^{12} (p_i - q_i) c_i \\
& + .50 * \max\left\{0, \sum_{i=1}^{12} (q_i - p_i) g_i - \max\left(1000, .25 \sum_{i=1}^{12} p_i g_i\right)\right\} \\
& + \max\left[0, \sum_{i=1}^{12} (p_i - x)\right] s_{yz}
\end{aligned}
\qquad (4.1)
$$

A corporation which knows its tax liability for the year with certainty would choose a payment path to minimize this loss. However, as a corporation's tax liability for the year is generally uncertain at the payment dates, determining the amount to pay in instalments each month is a problem involving decision making under uncertainty.

This uncertainty in a corporation's tax liability for a particular tax year is not constant over time. As economic events are observed, the corporation is better able to assess the profitability of the firm for the year and is therefore better able to estimate its tax liability. To determine how corporations should make instalment payments, it is therefore necessary to model this evolution of uncertainty over time.

Let there be 13 "information dates" at which all financial information about the firm is observed; the 12 payment dates (the last date of each month) and the remainder due date. Any possible complete history of the financial information about the corporation from date 1 to date 13 is a state of nature and is denoted by ω.[42] For each state of nature, there is an associated value of the monthly-average tax liability for the year, x. These values need not be uniquely associated with states of nature, *i.e.*, two or more states of nature may have the same monthly-average tax liability for the year. It is assumed that the corporation has perfect foresight regarding interest rates (both C_i and G_i). See chapter 5 for a discussion that interest rates, both C_i and G_i, may be uncertain.

Let an event be a subset of Ω, the set of all states of nature. A partition of Ω is a collection of events such that the union of these events is equal to Ω and the pairwise intersection of these events is empty. A given partition of Ω is said to be "finer" than another

---

[42]Notation for the information structure corresponds to that in Huang and Litzenberger [1988].

if any event in the latter partition is either an event in the former or a union of some events in the former.

For any corporation, let there exist an "information structure" $\mathbf{F} = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{13}\}$ where each $\mathcal{F}_i$ is a partition of Ω and has the property that $\mathcal{F}_t$ is finer than $\mathcal{F}_s$ if time $t > s$. This information structure is the formal representation of the process of information revelation through time. It is assumed that $\mathcal{F}_1 = \{\Omega\}$ and $\mathcal{F}_{13}$ is the partition of Ω generated by all of the individual states. It follows from these assumptions that the corporation knows at January 31 that the true state is in Ω, and it knows the identity of this true state at the remainder due date. To further clarify this notation, a simplified information structure is presented in Table 4.1.

TABLE 4.1

A Simple Information Structure

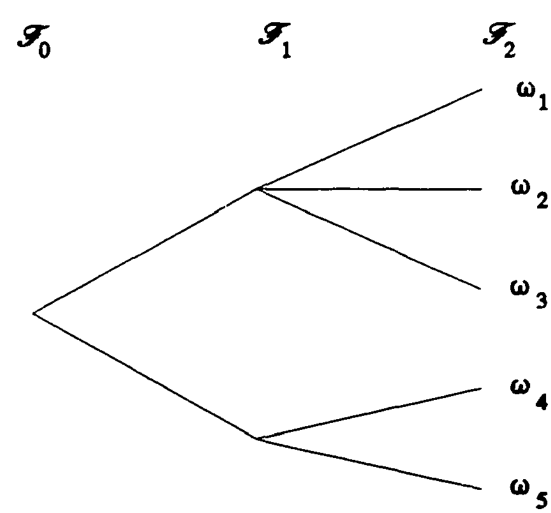
Consider a three period model (t = 0,1,2) with five states of nature ($\omega_1,\omega_2,\omega_3,\omega_4,\omega_5$). One possible information structure is as follows:

$\mathcal{F}_0 = \{\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}\}$
At date 0 it is known that five states of nature are possible: $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5$.

$\mathcal{F}_1 = \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}\}$
At date 1, the corporation is either in event 1 or event 2. In event 1 the possible states of nature are $\{\omega_1, \omega_2, \omega_3\}$. In event 2 the possible states of the nature are $\{\omega_4, \omega_5\}$.

$\mathcal{F}_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}\}$
At date 2 the true state of nature is known.

Note that $\mathcal{F}_2$ is finer than $\mathcal{F}_1$ which is finer than $\mathcal{F}_0$. This information structure can be represented diagrammatically as follows:



Source: Huang and Litzenberger [1988].

Corresponding to each information date is a set of events. Corresponding to each event at information dates 1 through 12 is a contingent payment p_{ij}, which is the amount paid at date i in event j. Let us denote J_i as the number of events belonging to $\mathcal{F}_i$; therefore, there are J_i endogenous variables in the model for date i. Note that $J_1 = 1$, as there is only one event at date 1, *i.e.*, all states of nature are possible. The total number of endogenous variables in the model is therefore $\sum_{i=1}^{12} J_i = K$. The amount paid at date 13 is simply the balance owing, if any, which is determined by equation (2.19) in chapter 2. This payment is not treated as an endogenous variable in this model. With the introduction of uncertainty, the vector p is henceforth, $p = [p_{1,1}, p_{2,1}, p_{2,2}, \dots, p_{2,J_2}, \dots, p_{12,1}, p_{12,2}, \dots, p_{12,J_{12}}]$. To reduce notational complexity, in some situations p is indexed by a single subscript k; *i.e.*, $p = [p_1, p_2, \dots, p_{k-1}, p_k, p_{k+1}, \dots, p_K]$.

To complete the information structure, let the probability at date 1 of each state ω be $Prob_\omega$ where $\sum_{\omega \in \Omega} Prob_\omega = 1$. Let these probabilities be known to the corporation at that date.

It is assumed in this thesis that the corporation is risk neutral. Given this assumption, its problem is to minimize its expected loss from paying instalments,

$$L(p) = \sum_{\omega \in \Omega} l(p^\omega; x^\omega)\, Prob_\omega \tag{4.2}$$

where: p^ω is the 12-element sub-vector of p which relates to that particular ω,

$$p^{\omega} = \left[p_1^{\omega}, p_2^{\omega}, \ldots, p_{12}^{\omega}\right]$$

where the subscript refers to the payments for each of the 12 months; and x^{ω} is the value of the monthly-average tax liability corresponding to the state of nature ω (*i.e.*, if at date i ω is a member of the jth event, then the ith element of p^{ω} is p_{ij}). That is, since any state of nature, ω, is a member of exactly one event for any date, it is associated with a vector of 12 contingent payments.

Substituting $l(p;x)$ from equation (4.1) above into equation (4.2), and noting that corresponding to each value of x^{ω} there exists an instalment liability for month i, q_i^{ω} (from equation (2.8) in chapter 2), the expected loss is;

$$\begin{aligned} L(p) = \sum_{\omega=1}^{\Omega} \Bigg\{ & \max\left[0, \sum_{i=1}^{12} (q_i^{\omega} - p_{ij}) g_i\right] + \sum_{i=1}^{12} (p_{ij} - q_i^{\omega}) c_i \\ & + .50 * \max\left[0, \sum_{i=1}^{12} (q_i^{\omega} - p_{ij}) g_i - \max\left(1000, .25 \sum_{i=1}^{12} q_i^{\omega} g_i\right)\right] \\ & + \max\left[0, \sum_{i=1}^{12} \left(p_{ij} - x^{\omega}\right) s_{yz}\right] \Bigg\} Prob_{\omega} \end{aligned} \tag{4.3}$$

where

$$q_i^{\omega} = \begin{cases} x^{\omega} & \forall\ i = 1 \text{ to } 12 \qquad \text{if } x^{\omega} \le b_1 \\ b_1 & \forall\ i = 1 \text{ to } 12 \qquad \text{if } b_1 \le \{x^{\omega}, b_2\} \\ \begin{cases} b_2 & \forall\ i = 1,2 \\ \frac{1}{10}(12 b_1 - 2 b_2) & \forall\ i = 3 \text{ to } 12 \end{cases} & \text{if } b_2 \le b_1 \le x^{\omega} \end{cases} \tag{4.4}$$

4.2 Calculating Derivatives of the Objective Function

The objective function in equation (4.3) above is non-differentiable. Hence, to state calculus-based conditions for an optimum of this function, it is necessary to define an appropriate derivative concept and a set of rules for calculating this derivative.

The concept of one-sided directional derivatives, which is discussed above in Appendix B in the context of a one-instalment model, is useful again in this multi-instalment context. Define a one sided directional derivative of $L(p)$ in the direction v at the point p_0 as,

$$L'(p_0; v) = \lim_{t \to 0^+} \frac{L(p_0 + tv) - L(p_0)}{t} \tag{4.5}$$

where $t > 0$ is a scalar and $t \to 0^+$ indicates that t approaches zero through the positive numbers. Note that v is of the same dimension as p; *i.e.*, the number of contingent payments. Recall that this dimension is denoted K (where $K = \sum_{i=1}^{12} J_i$), with individual elements denoted v_k. In other words, v specifies the direction of change for every contingent payment. Note that many elements v_k may occur at any date i; an element v_k corresponds to a particular event at a particular date.

Intuitively, v defines which contingent payments are to be increased or decreased, and in what relative amounts. The scalar t reflects the absolute magnitude of these increases/decreases. Therefore, $p_0 + tv$ represents the new payment vector created from an initial payment vector p_0 by increases/decreases in directions v by magnitude t.

Rewrite equation (4.3) as follows,

$$L(p) = \sum_{\omega \in \Omega} Prob_{\omega} \sum_{n=1}^{4} f_{\omega n}(p) \tag{4.6}$$

where,

$$f_{\omega 1}(p) = \max\left[0,\ I_{\omega}(p)\right] \tag{4.7}$$

$$f_{\omega 2}(p) = \sum_{i=1}^{12} \left(p_{ij} - q_i^{\omega}\right) c_i \tag{4.8}$$

$$f_{\omega 3}(p) = \max\left[0,\ Pen_{\omega}(p)\right] \tag{4.9}$$

$$f_{\omega 4}(p) = \max\left[0,\ S_{\omega}(p)\right] \tag{4.10}$$

and,

$$I_{\omega}(p) = \sum_{i=1}^{12} \left(q_i^{\omega} - p_{ij}\right) g_i \tag{4.11}$$

$$Pen_{\omega}(p) = .5 * \left[\sum_{i=1}^{12} \left(q_i^{\omega} - p_{ij}\right) g_i \ - \max\left(1{,}000,\ .25 \sum_{i=1}^{12} q_i^{\omega} g_i\right)\right] \tag{4.12}$$

$$S_{\omega}(p) = \sum_{i=1}^{12} \left(p_{ij} - x^{\omega}\right) s_{yz} \tag{4.13}$$

Applying the definition from equation (4.5) to the definition of $L(p)$ in equation (4.6),

$$
\begin{aligned}
L'(p_0; v) &= \lim_{t \to 0^+} \frac{\sum_{\omega \in \Omega} Prob_\omega \cdot \sum_{n=1}^{4} f_{\omega n}(p_0 + tv) - \sum_{\omega \in \Omega} Prob_\omega \cdot \sum_{n=1}^{4} f_{\omega n}(p_0)}{t} \\
&= \lim_{t \to 0^+} \frac{\sum_{\omega \in \Omega} Prob_\omega \cdot \sum_{n=1}^{4} \left[f_{\omega n}(p_0 + tv) - f_{\omega n}(p_0) \right]}{t} \\
&= \sum_{\omega \in \Omega} Prob_\omega \cdot \sum_{i=1}^{4} \lim_{t \to 0^+} \frac{f_{\omega n}(p_0 + tv) - f_{\omega n}(p_0)}{t} \\
&= \sum_{\omega \in \Omega} Prob_\omega \cdot \sum_{n=1}^{4} f'_{\omega n}(p_0; v)
\end{aligned} \tag{4.14}
$$

Thus, the one-sided directional derivative of the objective function is a probability-weighted sum of the one-sided directional derivatives of the components of the objective function relating to each state of nature. The one-sided directional derivative of each of these components is the sum of the one-sided directional derivatives of the four functions, $f_{\omega 1}$, $f_{\omega 2}$, $f_{\omega 3}$, and $f_{\omega 4}$.

The next step is to calculate the one sided directional derivative of each of the four functions. Consider first the function $f_{\omega 1}(p) = \max(0, I_\omega)$. Let us evaluate $f'_{\omega 1}(p_1)$ separately for:

$$
\begin{aligned}
&(a) \quad I_\omega > 0 \\
&(b) \quad I_\omega < 0 \\
&(c) \quad I_\omega = 0
\end{aligned}
$$

In relationship (a), the corporation is being charged instalment interest by the government, and hence the corporation is in an "underpayment" position. In contrast, relationship (c) arises where the corporation is in an "overpayment" position. Relationship (*b*) represents a knife-edge payment position between underpayment and overpayment.

(a) For values of p_0 such that $I_\omega(p_0) > 0,\ f_{\omega 1}(p_0) = I_\omega$. Therefore, for these values of p_0, f_ω is a differentiable function and, by Lemma 1 of Macnaughton [1993],

$$\begin{aligned} f'_{\omega 1}(p_0; v) &= v^T \nabla f_{\omega 1}(p_0; v) \\ &= \sum_{k=1}^{K} v_k \frac{\partial I_\omega}{\partial p_k} \end{aligned} \tag{4.16}$$

Since from equations (4.7) and (4.11) $\frac{\partial I_\omega}{\partial p_k} = -g_i$, it follows that,[43]

$$f'_{\omega 1}(p_0; v) = \sum_{k=1}^{K} v_k \,(-g_i) \tag{4.17}$$

(b) Similarly, for values of p_0 such that $I_\omega(p_0) < 0,\ f_{\omega 1}(p_0) = 0$ and,

$$\begin{aligned} f'_{\omega 1}(p_0; v) &= v^T \nabla f_{\omega 1}(p_0; v) \\ &= \sum_{k=1}^{K} v_k \frac{\partial (0)}{\partial p_k} \\ &= 0 \end{aligned} \tag{4.18}$$

[43]To clarify the relation of the subscripts k and i, recall that two different subscripting systems are used for the vector p: a single subscript k and a double subscript ij. Hence, p_k may be rewritten p_{ij}, and hence $\partial I_\omega / \partial p_{ij} = -g_i$.

(c) Now consider values of p_0 such that $I_\omega(p_0) = 0$. Clearly, $f_{\omega 1}$ is not differentiable at this point. However, for directions v which increase the value of this function, $f_{\omega 1}(p_0) = I_\omega$, which is a differentiable function. Therefore, applying Lemma 2 of Macnaughton [1993], the analysis of part (*a*) above applies and the derivative is the same as equation (4.17):

$$f'_{\omega 1}(p_0; v) = \sum_{k=1}^{K} v_k \left(-g_i\right) \tag{4.19}$$

Similarly, for values of p_0 such that $I_\omega = 0$ combined with directions v such that I_ω decreases or stays constant, $f_{\omega 1}(p_0) = 0$ and equation (4.18) applies;

$$f'_{\omega 1}(p_0; v) = 0 \tag{4.20}$$

Combining the results of parts (*a*), (*b*), and (*c*) above, it follows that,

$$f'_{\omega 1}(p_0; v) = \sum_{k=1}^{K} v_k \, I_{i\omega}(I_\omega, v) \tag{4.21}$$

where

$$I_{i\omega}(I_\omega, v) = \begin{cases} -g_i & \text{if } I_\omega > 0 \\ 0 & \text{if } I_\omega < 0 \\ -g_i & \text{if } I_\omega = 0 \quad \text{and } I_\omega \text{ increases in that direction} \\ 0 & \text{if } I_\omega = 0 \quad \text{and } I_\omega \text{ decreases or stays constant in that direction} \end{cases} \tag{4.22}$$

Using the same approach for $f_{\omega 3}$ and $f_{\omega 4}$ as for $f_{\omega 1}$, it follows that,

$$f'_{\omega 3}(p_0; v) = \sum_{k=1}^{K} v_k \, P_{i\omega}(Pen_\omega, v) \tag{4.23}$$

and

$$f'_{\omega 4}(p_0; v) = \sum_{k=1}^{K} v_k \, S_{i\omega}(S_\omega, v) \tag{4.24}$$

where

$$Pen_{i\omega}(Pen_\omega, v) = \begin{cases} -.5\, g_i & \text{if } Pen_\omega > 0 \\ 0 & \text{if } Pen_\omega < 0 \\ -.5\, g_i & \text{if } Pen_\omega = 0 \quad \text{and } P_\omega \text{ increases in that direction} \\ 0 & \text{if } Pen_\omega = 0 \quad \text{and } P_\omega \text{ decreases or stays constant in that direction} \end{cases} \tag{4.25}$$

and

$$S_{i\omega}(S_\omega, v) = \begin{cases} s_{yz} & \text{if } S_\omega > 0 \\ 0 & \text{if } S_\omega < 0 \\ s_{yz} & \text{if } S_\omega = 0 \quad \text{and } S_\omega \text{ increases in that direction} \\ 0 & \text{if } S_\omega = 0 \quad \text{and } S_\omega \text{ decreases or stays constant in that direction} \end{cases} \tag{4.26}$$

Note that $S_{i\omega}$ is not dependant on the date; *i.e.*, it is independent of i. In other words, for any given value of S_{ω}, the value of $S_{i\omega}$ is the same regardless of which payment is being changed.

Finally, $f_{\omega 2}(p_0)$ is a differentiable function for all values of p_0. It follows from Lemma 1 of Macnaughton [1993] that,

$$\begin{aligned} f'_{\omega 2}(p_0; v) &= v^T \nabla f_{\omega 2}(p_0; v) \\ &= \sum_{k=1}^{K} v_k\, c_i \end{aligned} \tag{4.27}$$

Substituting equations (4.21), (4.23), (4.24), and (4.27) into equation (4.14) gives a useful definition of $L'(p_0, v)$:

$$L'(p_0; v) = \sum_{\omega \in \Omega} Prob_{\omega} \sum_{k=1}^{K} v_k \cdot \left(I_{i\omega} + c_i + Pen_{i\omega} + S_{i\omega}\right) \tag{4.28}$$

where the constituent terms $I_{i\omega}$, $P_{i\omega}$, and $S_{i\omega}$ are defined in equations (4.22), (4.25), and (4.26) respectively. This rule for calculating the one-sided directional derivative of the objective function is used extensively below.

4.3 Conditions for an Optimum

From Diewert [1981][44] and Macnaughton [1993], any optimal solution p_0 to the problem

$$\underset{p}{\textit{minimize}} \; L(p) \qquad \textit{subject to } p \geq 0 \tag{4.29}$$

must satisfy the Diewert condition,

$$L'(p_0; v) \geq 0 \tag{4.30}$$

for all feasible directions v since the function $L(p)$ is finite-valued and one-sided directionally differentiable. If this were not true, there would exist a point $p_0 + tv$ which has a lower objective function value. In other words, a corporation could reduce its expected loss through increasing or decreasing certain payments. Thus, the Diewert condition is a necessary condition for an optimum.

Furthermore, since the constraint functions are linear and as $L(p)$ is convex, as proven in Appendix F, the results of Diewert [1981] and Macnaughton [1993] imply that any point p_0 which satisfies the Diewert condition is sufficient for an optimum; *i.e.*, a local optimum which is not also a global optimum cannot exist.

[44]See theorems 20 and 21.

4.4 Optimal Payment Structure with Simple and Constant Rates of Interest and Without Stub Loss

In this section three additional assumptions are made: first, the corporation cannot suffer a stub loss, *i.e.*, $f_{\omega 4}(p) = 0$; second, both government and private interest rates are simple (are not compounded);[45] and third, the rates are constant across payment dates. These assumptions are made to isolate the effect of the evolution of uncertainty on a corporation's payment strategy and to simplify analysis. The first two assumptions are relaxed in part in sections 4.5 and 4.6 below.

To determine the optimal payment strategy for this problem, it is convenient to begin by assuming that all instalment payments prior to the final month of the fiscal year are zero. Let us denote any payment vector satisfying this restriction as $\tilde{p}$; that is,

$$\tilde{p}_{ij} = 0 \quad \forall\ i < 12 \tag{4.31}$$

Event j of date 12 may be partitioned into 5 sub-events Ω_1 to Ω_5:[46]

[45]The assumption that rates are simple is counterfactual (as are the other two assumptions); recall that the rate g_i is compounded daily. The rates g_i and c_i in equations (2.10) and (2.14) of chapter 2 may be rewritten as constant non-compounded rates,

$$g_i = G\,\frac{\sum_{j=i+1}^{13} N_j}{365} \quad and \quad c_i = C\,\frac{\sum_{j=i+1}^{13} N_j}{365}$$

The effects of compounding are small (as is demonstrated in section 4.6 below).

[46]This is a partition of event j as $Pen_\omega \geq 0 \rightarrow I_\omega > 0$ (see Lemma 4.7 in Appendix F).

$$\Omega_1 = \{ \omega \mid Pen_\omega > 0, I_\omega > 0 \} \tag{4.32}$$

$$\Omega_2 = \{ \omega \mid Pen_\omega = 0, I_\omega > 0 \} \tag{4.33}$$

$$\Omega_3 = \{ \omega \mid Pen_\omega < 0, I_\omega > 0 \} \tag{4.34}$$

$$\Omega_4 = \{ \omega \mid Pen_\omega < 0, I_\omega = 0 \} \tag{4.35}$$

$$\Omega_5 = \{ \omega \mid Pen_\omega < 0, I_\omega < 0 \} \tag{4.36}$$

with corresponding probabilities,

$$prob_l = \sum_{\omega \in \Omega_l} Prob_\omega, \quad l = 1,2,3,4,5 \tag{4.37}$$

Note that the classification given above is a function of the payment amount $\bar{p}_{12,j}$. For example, increasing this payment would reduce the value of Pen_ω and I_ω and would therefore tend to increase $prob_1$ and decrease $prob_5$.

Note further that there is a different partition for each event $j = 1$ to J_{12}. That is, a partition of event j of date 12 is different from the partition of event $k \neq j$ of date 12 as different states of nature comprise that particular event.

Lemma 4.1:

The necessary and sufficient conditions for an optimum to the problem of minimizing $L(p)$ in equation (4.3) subject to the 3 assumptions listed above and the restriction that all non-date 12 payments are zero are that the amount of each contingent date 12 payment $\tilde{p}_{12,j}$ satisfies:

$$\begin{cases} 0 \le \text{common terms} + prob_2(-g_{12} + c_{12}) + prob_4 c_{12} & \text{if } \tilde{p}_{12,j} = 0 \quad (4.38) \\ \text{common terms} + prob_2(-1.5g_{12} + c_{12}) + prob_4(-g_{12} + c_{12}) & \\ \qquad \le 0 \le & \\ \text{common terms} + prob_2(-g_{12} + c_{12}) + prob_4 c_{12} & \text{if } \tilde{p}_{12,j} > 0 \quad (4.39) \end{cases}$$

where

$$\text{common terms} = prob_1(-1.5g_{12} + c_{12}) + prob_3(-g_{12} + c_{12}) + prob_5 c_{12}$$

The proof of this Lemma is in Appendix F.

To aid in the interpretation of equation (4.39), Figure 4.1 provides an illustration of the corporation's loss and expected loss where, at date 12, the corporation knows that the true state of nature is either ω_1 or ω_2 with equal probability: $Prob_{\omega_1} = Prob_{\omega_2} = .5$.

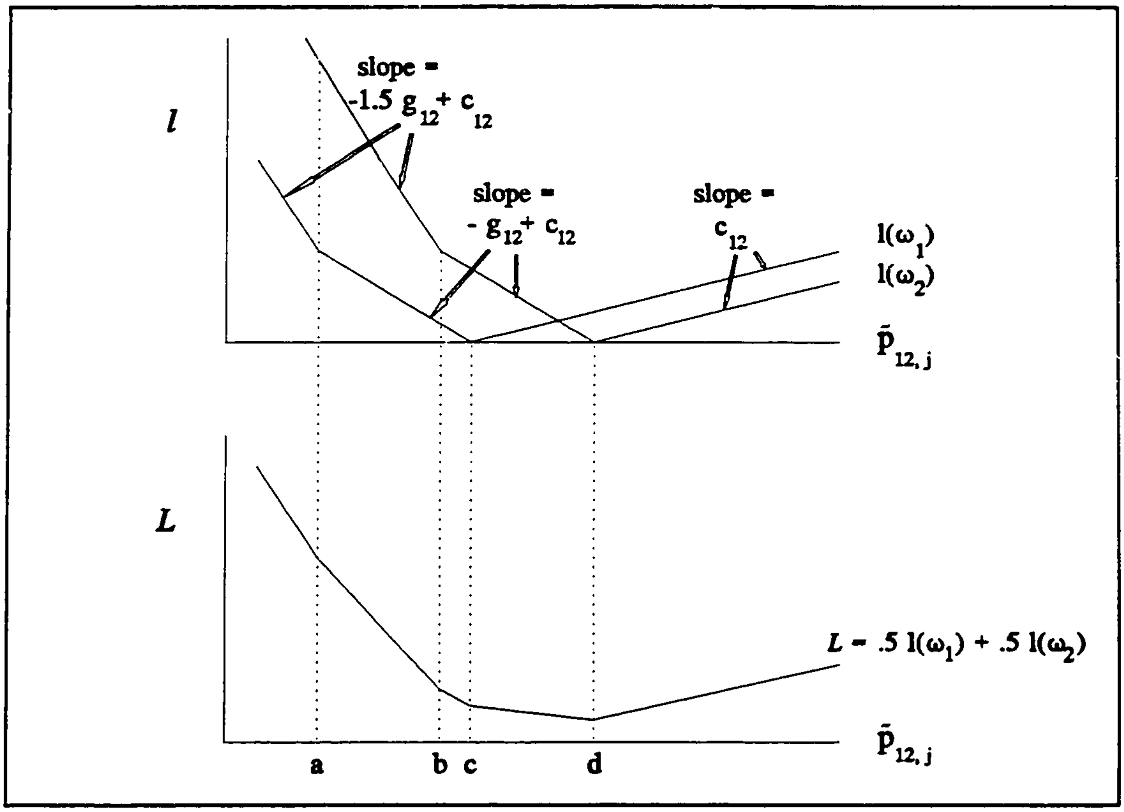


FIGURE 4.1
Kink Structure for the Expected Loss

Let us examine the top panel of Figure 4.1, which depicts the loss for states of nature ω_1 and ω_2. Consider first the slopes of the three linear sections of each loss moving from left to right.

(1). If a corporation has underpaid such that it incurs a penalty, it will owe instalment interest and the penalty. Hence, if it increases the amount it pays at date 12 by \$1, it will decrease instalment interest by g_{12} and the penalty by $.5g_{12}$. The increased payment of \$1 at date 12 also has an opportunity cost to the corporation of c_{12}.

Therefore, the slope in this penalty region is $-1.5g_{12} + c_{12}$.

(2). If the corporation makes a date 12 payment such that it incurs instalment interest but no penalty, increasing that payment by $1 will decrease instalment interest by g_{12}, but with an associated opportunity loss of c_{12}. The slope in this region is therefore $-g_{12} + c_{12}$.

(3). If the corporation makes a date 12 payment such that it has overpaid with respect to instalment interest, the cost of increasing the date 12 payment by $1 will be the opportunity loss c_{12}.

Note that kinks exist where these line segments join; at $Pen_\omega = 0$ (at a payment amount where the corporation does not incur a penalty, but would if it paid $1 less) and at $I_\omega = 0$ (where the corporation neither overpays nor underpays). In Figure 4.1, the period 12 payment amounts a and b are the value of $\tilde{p}_{12,j}$ where $Pen_{\omega_1} = 0$ *and* $Pen_{\omega_2} = 0$ respectively, and c and d represent the value of $\tilde{p}_{12,j}$ where $I_{\omega_1} = 0$ *and* $I_{\omega_2} = 0$ respectively.

Let us now discuss the bottom panel of Figure 4.1. The expected loss, L, is a weighted sum of the loss for ω_1 and for ω_2; in this problem the losses are equally weighted as $Prob_{\omega_1} = Prob_{\omega_2} = .5$. That is, the expected loss is determined through adding the losses for each date 12 payment amount and dividing by two. The date 12 payment amount d is the optimum.

That d is the optimal payment amount may also be seen through the algebra in equation (4.39). At that point, there is an overpayment with respect to instalment interest in state of nature 1, so $\omega_1 \in \Omega_5$. In state of nature 2, there is no underpayment or overpayment with respect to instalment interest, so $\omega_1 \in \Omega_4$. Hence, $prob_5 = prob_4 = .5$, and $prob_1 = prob_2 = prob_3 = 0$. Therefore equation (4.39) reduces to $.5(-1.5g_{12}+c_{12})+.5c_{12} \le 0 \le .5c_{12}+.5c_{12}$, which is satisfied.

The result in conditions (4.38) and (4.39) can be interpreted in terms of the slopes of the expected loss function to the left and right of any particular point. Let $slope_L$ and $slope_R$ denote these left and right slopes respectively. Then these conditions can be rewritten as:[47]

$$\begin{cases} 0 \le slope_R & \text{if } \bar{p}_{12,j} = 0 \\ slope_L \le 0 \le slope_R & \text{if } \bar{p}_{12,j} > 0 \end{cases} \tag{4.40}$$

As the left and right slopes are only unequal at a kink, an optimum is likely to occur at a kink.

As the loss for any particular state of nature is continuous and piecewise linear, and as the slopes are everywhere non-zero[48], the minimum value for this loss is at a kink or is a boundary solution, $\bar{p}_{12,j} = 0$. As the expected loss is a simple weighted sum of these piecewise linear losses, it too is piecewise linear, and will normally have a kink at each payment amount coinciding with a kink in a particular state of nature. For example, in Figure 4.1 the date 12 payment amount which minimizes the corporation's expected loss is *d*: this is

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[47]Furthermore, each of these one-sided slopes can be written as a probability-weighted sum of the one-sided slopes of the individual loss amounts *l*. For example, the right-side slope of the expected loss ($slope_R$ above) is the following probability weighted sum of right-side slopes of loss amounts,

$$prob_1(-1.5g_{12}+c_{12}) + (prob_2+prob_3)(-g_{12}+c_{12}) + (prob_4+prob_5)c_{12}$$

[48]The slope of the loss cannot be zero in any region as it is assumed that $G_i > C_i > 0$, which implies that $g_i > c_i > 0$.

the amount associated with neither underpaying nor overpaying with respect to instalment interest in the year in the state of nature ω_2 ($I_\omega = 0$). Note, however, that it is possible for a particular set of parameters and probabilities (through coincidence), that a flat section may exist in the expected loss.[49]

One aspect of Lemma 4.1 which is not illustrated in Figure 4.1 is the relationship of the corporation's loss to the first instalment base. To examine this, consider a case in which there are 5 possible states of nature, each with 20% probability of occurring. These states of nature differ in their monthly-average tax liability as follows:

$$\begin{aligned} \omega_1&: x = .8\,b_1 \\ \omega_2&: x = .9\,b_1 \\ \omega_3&: x = 1.0\,b_1 \\ \omega_4&: x = 1.1\,b_1 \\ \omega_5&: x = 1.2\,b_1 \end{aligned}$$

From equation (2.8) the associated values of the instalment liability are (assuming for simplicity that $b_2 < b_1$),

$$\begin{aligned} &\omega_1: q_i = .8\,b_1 \qquad i = 1,2,\cdots,12 \\ &\omega_2: q_i = .9\,b_1 \qquad i = 1,2,\cdots,12 \\ &\omega_3, \omega_4, \omega_5: q_i = 1.0\,b_1 \qquad i = 1,2,\cdots,12 \end{aligned}$$

as the instalment liability for any payment date cannot exceed the monthly-average tax liability.

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[49]In the above example a flat section in the expected loss, where the derivative equals zero for a finite length, would exist if $g_{12} = 2c_{12}$.

Hence, the value of the loss for ω_3, ω_4, and ω_5 is identical for any payment amount. The possible values of the loss function and associated probabilities are: $l(\omega_1)$, 20%; $l(\omega_2)$, 20%; $l(\omega_3,\omega_4,\omega_5)$, 60%. Hence, the loss associated with ω_3, ω_4, and ω_5 receives disproportionate weighting in the expected loss and therefore the minimum of the expected loss is likely to be at one of the kinks at the loss function associated with ω_3, ω_4, and ω_5.

These points are illustrated in Figure 4.2 below. In the top panel, there are three losses shown: $l(\omega_1)$; $l(\omega_2)$; and $l(\omega_3,\omega_4,\omega_5)$. As the tax liability increases, the loss function shifts to the right until the loss associated with the first instalment base (shown in bold) is reached; for

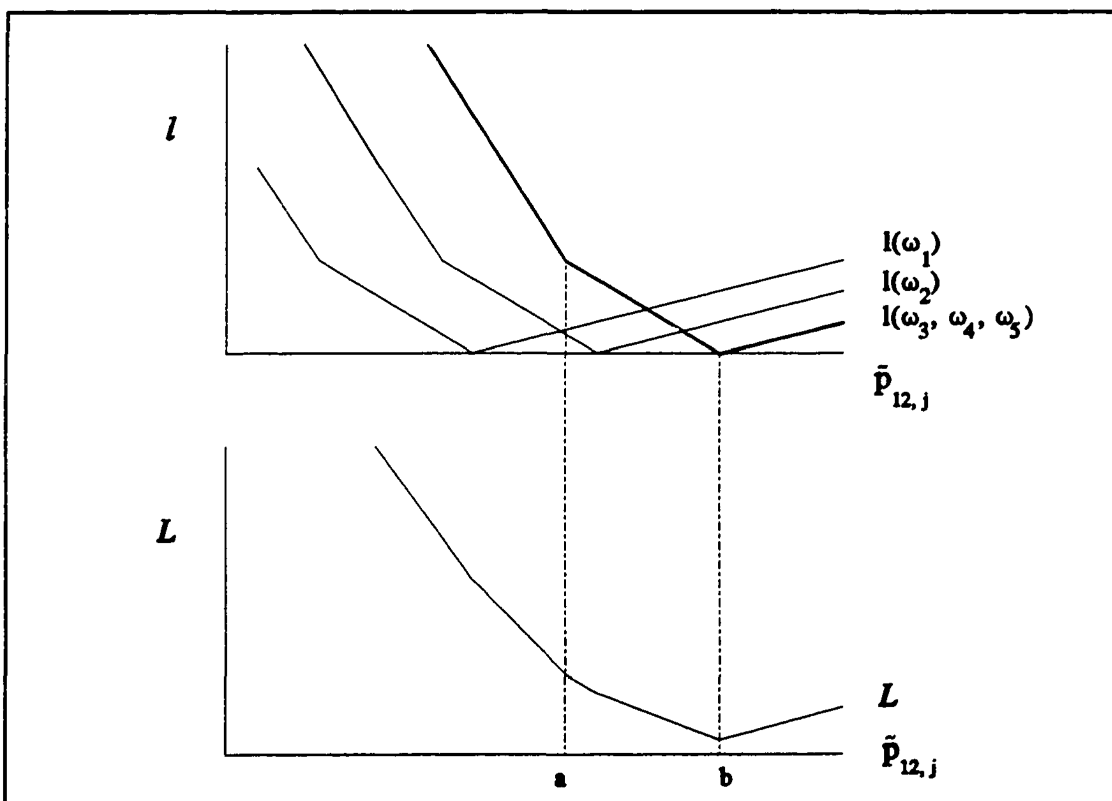

FIGURE 4.2
Relationship of Kink Structure to the First Instalment Base

all tax liabilities greater than or equal to the first instalment base, the bolded loss function is the appropriate loss function. Hence, greater probability is placed on that loss function than on either of the other two loss functions. The bottom panel demonstrates the effect of this disproportionate weighting on that expected loss. The two kinks of this right most loss function are likely to be "sharper" than other kinks and hence are more likely to be the minimum of the expected loss. The first of these two kinks occur where a penalty becomes impossible for any conceivable state of nature, *i.e.*, $Pen_\omega = 0$ for ω such that $x \geq b_1$. Algebraically this is,

$$\tilde{p}_{12,j} = \min\left(.75\frac{\sum_{i=1}^{12} b_1 g_i}{g_{12}}, \frac{\sum_{i=1}^{12} b_1 g_i - 1000}{g_{12}}\right) \qquad (4.41)$$

which is point *a* on the graph. The second sharp kink occurs where an overpayment becomes impossible for any conceivable sate of nature, *i.e.*, $I_\omega = 0$ for ω such that $x \geq b_1$. Algebraically this is,

$$\tilde{p}_{12,j} = \frac{\sum_{i=1}^{12} b_1 g_i}{g_{12}} \qquad (4.42)$$

which is point *b* on the graph. In this example, point *b* is the minimum expected loss.

Let us summarize the possible optima to the problem of minimizing the expected loss subject to the restriction that all non-date 12 payments are zero. There are five possibilities:

$$(i) \quad \tilde{p}_{12,j} = 0$$

$$(ii) \quad 0 < \tilde{p}_{12,j} < \min\left(.75\frac{\sum_{i=1}^{12} b_1 g_i}{g_{12}}, \frac{\sum_{i=1}^{12} b_1 g_i - 1000}{g_{12}} \right)$$

$$(iii) \quad \tilde{p}_{12,j} = \min\left(.75\frac{\sum_{i=1}^{12} b_1 g_i}{g_{12}}, \frac{\sum_{i=1}^{12} b_1 g_i - 1000}{g_{12}} \right) \qquad (4.43)$$

$$(iv) \quad \min\left(.75\frac{\sum_{i=1}^{12} b_1 g_i}{g_{12}}, \frac{\sum_{i=1}^{12} b_1 g_i - 1000}{g_{12}} \right) < \tilde{p}_{12,j} < \frac{\sum_{i=1}^{12} b_1 g_i}{g_{12}}$$

$$(v) \quad \tilde{p}_{12,j} = \frac{\sum_{i=1}^{12} b_1 g_i}{g_{12}}$$

These five solutions exactly parallel the five graphs in Figure 3.5 of chapter 3. The principal difference is that solutions (*ii*) and (*iv*) above are also at kinks, although not at "sharp" kinks. These solutions will be at "shallow" kinks at which $Pen_\omega = 0$ or $I_\omega = 0$ for some particular state of nature such that the tax liability is less than the first instalment base.

The algebra defining the sharp kinks is very similar between the single-period model in chapter 3 and the multi-period model in this section. The payment amount at which a penalty becomes impossible for any state of nature is,

$$p^* = \min\left(.75 B_1,\ B_1 - \frac{1000}{g}\right)$$

in chapter 3, and

$$\bar{p}_{12,j} = \min\left(.75\,\frac{\sum_{i=1}^{12} b_1 g_i}{g_{12}},\ \frac{\sum_{i=1}^{12} b_1 g_i - 1000}{g_{12}}\right)$$

in this section.

Similarly, the point at which overpayment becomes impossible for any state of nature is,

$$p^* = B_1$$

in chapter 3, and

$$\bar{p}_{12,j} = \frac{\sum_{i=1}^{12} b_1 g_i}{g_{12}}$$

in this section.

It is also possible to draw a close parallel between the optimality conditions in chapter 3 and in this section. Consider the payment amount at which a penalty becomes impossible for all states of nature. In this section, the condition from equation (4.39) is,

$$\begin{aligned} prob_2(-1.5g_{12} + c_{12}) + (prob_3 + prob_4)(-g_{12} + c_{12}) + prob_5 c_{12} \\ \leq 0 \leq \\ (prob_2 + prob_3 + prob_4)(-g_{12} + c_{12}) + prob_5 c_{12} \end{aligned} \tag{4.44}$$

as it is not possible to have a penalty for this payment amount (*i.e.*, $prob_1 = 0$). To illustrate the parallel conditions from chapter 3, it is first necessary to interpret the integrals for tax liability in the terms used in this chapter:

$$\int_{B_1}^{\infty} f(X)\,dX = prob_2 \qquad (Pen_\omega = 0)$$

$$\int_{.75B_1}^{\infty} f(X)\,dX = prob_2 + prob_3 \qquad (I_\omega = 0)$$

$$\int_{0}^{.75B_1} f(X)\,dX = prob_5 \qquad (I_\omega < 0)$$

Substituting these expressions into equation (3.35), the optimal conditions for chapter 3 are,[50]

$$prob_2(-1.5g_{12} + c_{12}) + prob_3(-g_{12} + c_{12}) + prob_5 c_{12}$$
$$\leq 0 \leq$$
$$(prob_2 + prob_3)(-g_{12} + c_{12}) + prob_5 c_{12}$$

which is the identical to equation (4.44) above given that $prob_4 = 0$ in the continuous model as the density of tax liability at any single point is zero.

As a second illustration of the parallel in optimality conditions between chapter 3 and this section, consider the point at which an overpayment becomes impossible for any state of nature. In this chapter, the condition from (4.39) is:

$$prob_4(-g_{12} + c_{12}) + prob_5 c_{12} \leq 0 \leq (prob_4 + prob_5) c_{12} \quad (4.45)$$

as underpayment is impossible at this payment amount (*i.e.*, $prob_1$, $prob_2$, $prob_3 = 0$). To illustrate the parallel conditions from chapter 3, let us again reinterpret certain integrals from

[50]This analysis concerns conditions (3.35) which assumes that $B_1 > 4000/g$. Similar analysis may be used for conditions (3.34) where $B_1 < 4000/g$.

chapter 3:

$$\int_{B_1}^{\infty} f(X)\,dX = prob_4$$

$$\int_{0}^{B_1} f(X)\,dX = prob_5$$

Substituting these expressions into equation (3.35) yields,

$$prob_4\,(-g_{12} + c_{12}) + prob_5\,c_{12}$$

which is the left-hand side of condition (4.45) above. The right-hand side of condition (4.45) above is provided in chapter 3 by the proof in section 3.2.1 that payments greater that B_I need not be considered because $c > 0$.

The following lemma shows that under the assumptions of this section there is no loss of generality in restricting all non-date 12 payments to be zero.

Lemma 4.2:

Where the corporation cannot suffer a stub loss and where rates are simple and non-stochastic, for any payment vector p, there exists a vector $\bar{p}$ defined by,

$$\bar{p}^{\omega} = \left[0,0,0,0,0,0,0,0,0,0,0, \frac{\sum_{i=1}^{12} p_i^{\omega} g_i}{g_{12}} \right] \tag{4.46}$$

where,

$$L(p) = L(\bar{p}) \tag{4.47}$$

and p^{ω}, the 12-element sub-vector of p which relates to a particular state of nature ω, is defined in the discussion of equation (4.2) above. In other words, for any payment vector in which one or more non-date 12 payments are positive, there exists a payment vector with all non-date 12 payments equal to zero which has the same value of the objective function. The proof of this Lemma is in Appendix F.

The implication of this result is that where a corporation cannot suffer a stub loss and rates are simple and non-stochastic, it can choose a payment amount at date 12 which would provide an identical expected loss to any particular payment path. That is, it cannot hurt the corporation to delay all payments until date 12.

We may now demonstrate the central result in this section that the restricted optimum in Lemma 4.1 above is an optimum for the general or unrestricted problem, although other solutions to the general problem may also exist.

Proposition 4.1:

Assume that the corporation cannot suffer a stub loss and that rates are simple, non-stochastic, and unchanging. For the problem of minimizing equation (4.3) subject to these assumptions, the necessary and sufficient conditions are,

$$p = \tilde{p}^*, \qquad or \qquad \frac{\sum_{i=1}^{12} p_i^{\omega} g_i}{g_{12}} = \tilde{p}_{12}^{\omega *} \quad \forall\ \omega \tag{4.48}$$

where $\tilde{p}^*$ is the optimum for the problem of minimizing equation (4.3) subject to the restriction that all non-date 12 payments are zero which is given in equations (4.38) and (4.39) above, and $\tilde{p}^{\omega *}$ is the 12-element sub-vector of $\tilde{p}^*$ which corresponds to a particular ω. Note that $p = \tilde{p}^*$ satisfies the right-hand side of equation (4.48), so the left-hand side of equation (4.48) could be omitted. This proposition is proven in Appendix F.

In certain circumstances it is not suboptimal to pay prior to date 12. The intuition is that a corporation may make positive non-date 12 payments if it would not increase the expected loss from instalments. A payment will increase a corporation's expected loss from instalments if, for any potential future date 12 event, the corporation would choose to make a negative payment if it were so allowed. An alternative way of expressing this is, if the corporation would pay zero at some future event, it would not make a positive payment now.

This notion is stated more specifically in the following proposition.

Proposition 4.2:

Consider event $\phi_{t,k}$ which is the event at date t which has history k. Because the partition at date 12 is finer than the partition at any date $t < 12$, it is possible to write this event as,

$$\phi_{tk} = \bigcup_{j=m}^{M} \phi_{12j} \tag{4.49}$$

In other words, if event k occurs at date t, then at date 12 the only possible events are $\phi_{12,m}, \phi_{12,m+1}, \cdots, \phi_{12,M}$. Then, for a vector p which satisfies equation (4.48) above,

$$p_{tk} \leq \frac{g_{12}}{g_t} \min\left\{\bar{p}^*_{12m}, \bar{p}^*_{12m+1}, \cdots, \bar{p}^*_{12M}\right\} \tag{4.50}$$

Proof:

Assume that equation (4.50) is not true. Then there exists an l where $m \leq l \leq M$, such that,

$$p_{tk} = \frac{g_{12}}{g_t} \bar{p}^*_{12l} + \epsilon \tag{4.51}$$

where $\epsilon > 0$. Let $p_{12,k}$ correspond to some state ω, *i.e.*,

$$p^{\omega} = \left[p_1^{\omega}, p_2^{\omega}, \cdots, p_{t-1}^{\omega}, p_{tk}, p_{t+1}^{\omega}, \cdots, p_{12}^{\omega}\right]$$

Substituting equation (4.51) into the right-hand equation in (4.48),

$$\sum_{i=1}^{t-1} g_i p_i^{\omega} + g_t \left[\frac{g_{12}}{g_t}\bar{p}_{12l}^{*} + \epsilon\right] + \sum_{i=t+1}^{12} g_i p_i^{\omega} = g_{12}\bar{p}_{12l}^{*} \qquad (4.52)$$

Rewriting this expression,

$$\sum_{i=1}^{t-1} g_i p_i^{\omega} + \sum_{i=t+1}^{12} g_i p_i^{\omega} = -g_t \epsilon$$

$$\Rightarrow \sum_{i=1}^{t-1} g_i p_i^{\omega} + \sum_{i=t+1}^{12} g_i p_i^{\omega} < 0 \quad \textit{since } \epsilon > 0 \textit{ and } g_i > 0 \qquad (4.53)$$

There is no p^{ω} vector which can satisfy this condition and still meet the non-negativity requirement, $p^{\omega} \geq 0$. Therefore, this is a contradiction.

Q.E.D.

It is worthwhile to briefly examine a special case of the above model; where the tax liability for the year is certain. The above formulation under uncertainty may be restricted to a certainty case through defining only a single state of nature which, by definition, has probability one (*i.e.*, $\Omega = \{\omega\}$).

Corollary 4.1:

Assume that a corporation's tax liability for the year is known with certainty, that it cannot suffer a stub loss, and that rates are simple, non-stochastic, and unchanging. For the problem of minimizing equation (4.3) subject to these assumptions, necessary and sufficient conditions are,

$$I_\omega = 0 \quad \textit{and} \quad p_{ij}^{\omega} = 0 \;\; \forall\, i < 12 \tag{4.54}$$

Thus, where the corporation's tax liability for the year is known with certainty, it in not optimal to underpay ($I_\omega > 0$) or to overpay ($I_\omega < 0$) with respect to instalment interest. Corollary 4.1 is proven in Appendix F.

4.5 Optimal Payment Structure with Stub Loss Under Certainty

This section develops a multi-payment certainty model (*i.e.*, the corporation's tax liability for the year is known at the beginning of its fiscal year) where rates are assumed simple and unchanging. It is worthwhile to develop this model for the following reasons: first, effects associated with the stub loss may be isolated; and second, the results are of direct interest to a class of corporations (those that know their instalment liability for the year).

The analytic framework defined above in section 4.2 simplifies to a single state (to simplify notation, the ω superscript is therefore omitted) with the complete set of endogenous variables $p_1, p_2, \ldots, p_{12}$. Note that the optimal payment structure as defined in the following proposition includes the certainty result developed, as a special case of the uncertain model without stub loss, in section 4.4.

Proposition 4.3:

Where the rates g_i and c_i are simple (not compounded) and non-stochastic, the necessary and sufficient conditions for an optimum are as follows:

$$\sum_{i=1}^{12} p_i g_i = \sum_{i=1}^{12} q_i g_i \tag{4.55}$$

and,

$$\sum_{i=1}^{12} p_i \leq \sum_{i=1}^{12} x_i \tag{4.56}$$

Proof of Sufficiency:

Substituting equation (4.55) into the definitions of $I_\omega(p)$ and $Pen_\omega(p)$, equations (4.11) and (4.12) respectively,

$$\begin{aligned} I_\omega &= 0 \\ Pen_\omega &< 0 \end{aligned} \tag{4.57}$$

which implies that $f_{\omega 1}$ (equation (4.7)) and $f_{\omega 3}$ (equation (4.9)) equal zero. Substituting equation (4.56) into the definition of S_ω in equation (4.13),

$$S_\omega \leq 0 \tag{4.58}$$

which implies that $f_{\omega 4}$ (equation (4.10)) equals zero. Further, from equation (4.55), $f_{\omega 2}$

(equation (4.8)) also equals zero.[51] Therefore, for any p_0 which satisfies equations (4.55) and (4.56),

[51]**Proof:** With simple and unchanging rates,

$$g_i = G \cdot \frac{\sum_{k=i+1}^{13} N_k}{365}, \quad and \; c_i = C \cdot \frac{\sum_{k=i+1}^{13} N_k}{365}$$

By equation (4.63),

$$\sum_{i=1}^{12} (p_i - q_i) g_i = 0$$
$$\Rightarrow \sum_{i=1}^{12} (p_i - q_i) G \cdot \sum_{k=i+1}^{13} \frac{N_k}{365} = 0$$
$$\Rightarrow \sum_{i=1}^{12} (p_i - q_i) \sum_{k=i+1}^{13} \frac{N_k}{365} = 0$$

It therefore follows that,

$$C \sum_{i=1}^{12} (p_i - q_i) \sum_{k=i+1}^{13} \frac{N_k}{365} = 0$$
$$\Rightarrow \sum_{i=1}^{12} (p_i - q_i) c_i = 0$$
$$\Rightarrow f_{\omega_2} = 0$$

$$L(p) = \sum_{\omega=1}^{\Omega} Prob_\omega \cdot \left(f_{\omega 1}(p) + f_{\omega 2}(p) + f_{\omega 3}(p) + f_{\omega 4}(p)\right)$$
$$= \sum_{\omega=1}^{\Omega} Prob_\omega \cdot 0 \qquad (4.59)$$
$$= 0$$

Also, the minimum value of the objective function is zero.[52]

Proof of Necessity:

Let us briefly sketch the nature of the proof, and leave the details for Appendix F. Where the rates g_i and c_i are simple and non-stochastic any time path of payments which does not satisfy equations (4.55) and (4.56) is not optimal. This is proven through demonstrating three relationships:

(*a*) where $I_\omega < 0$, there exists a direction v in which $L'(p_0; v) < 0$;

(*b*) where $I_\omega > 0$, there exists a direction v in which $L'(p_0; v) < 0$; and

(*c*) where $I_\omega = 0$ and $S_\omega > 0$, there exists a direction v in which $L'(p_0; v) < 0$.

---

[52]From equation (4.1), where $q_i = p_i$ for all $i = 1$ to 12, the first three terms (U, O, and *Pen*) equal zero. Where $p_i = q_i$ for all i, we can rewrite the last term as,

$$\max\left[0, \sum_{i=1}^{12} (q_i - x)\right]$$

It follows from the definition of q_i in chapter 2 that

$$\sum_{i=1}^{12} q_i \le \sum_{i=1}^{12} x,$$

and therefore the final term will also equal zero.

Relationship (*a*) states that where the corporation is in an overpayment position in the instalment period, it can reduce its expected loss. This may be shown through demonstrating that paying less at any date, for which it would make a payment, will reduce its expected loss. Relationship (*b*) states that where the corporation is in an underpayment position, it can reduce its expected loss. This may be proven through demonstrating that where it pays a positive amount at a non-January date, it can reduce its expected loss by shifting an amount made after January to January (by making payments earlier), and if it only makes a January payment, it may reduce its loss through increasing that payment. Relationship (*c*) states that where the corporation has neither overpaid nor underpaid in the instalment period, but has incurred a stub loss, it may reduce its expected loss; it may reduce its loss through shifting an amount from a non-January payment date to the January payment date (though paying earlier).

The intuition behind the necessary and sufficient conditions for an optimum (for this single state multi-period model), defined in equations (4.55) and (4.56) is as follows. First, any payment path which satisfies equation (4.56) will ensure that a stub loss cannot occur. That is, where this condition is met, the corporation will never pay more in the instalment period than its tax liability for the year. Equation (4.55), the other half of the optimality conditions, ensures that there is neither an underpayment nor an overpayment with respect to instalment interest.

An intuitive optimum for this problem is that a corporation pay an amount exactly equal to its instalment liability in each period. That is, paying $p_i = q_i \;\; \forall \; i = 1, 2, \ldots, 12$ satisfies equation (4.55) and (4.56). A second optimum, making a single payment at date 1 in the amount,

$$p_1 = \frac{\sum_{i=1}^{12} q_i g_i}{g_1} \tag{4.60}$$

is of interest as it provides the smallest payment amount in the instalment period and as it appears to be an optimum for the problem where rates are compounded (as is discussed in the following section).

4.6 Optimal Payment Structure with Compound Interest Under Certainty

The purpose of this section is to examine the effect of using compounded rates on the optimal solution. To isolate the effects, the simplifying assumptions utilized in section 4.5 are retained: that is, the corporation's tax liability for the year is known with certainty at the beginning of its fiscal year, and rates are non-stochastic and unchanging. Therefore, the assumptional base is the same as in section 4.5 except that rates are compounded.

Proposition 4.4:

Assume that the corporation's tax liability for the year is known with certainty at the beginning of its fiscal year, and that rates are non-stochastic. For the problem of minimizing equation (4.3) subject to these assumptions, the necessary and sufficient conditions for an optimum are,

$$p_1 = \frac{\sum_{i=1}^{12} q_i g_i}{g_1} \qquad (4.61)$$

$$p_j = 0, \quad j \geq 2$$

This proposition is proven in Appendix F.

This may initially appear surprising. Recall from chapter 2 that in the single instalment model under certainty it is optimal for a corporation to make an instalment payment which is equal to its liability. One may expect that making payments equal to one's instalment liability each month would similarly be optimal in a multi-instalment model; that is, the corporation could minimize its loss through paying $p_i = q_i$ for all $i = 1$ to 12.

Why can a corporation "do better" in following the payment strategy in equation (4.61) than in paying its instalment liability each period? The result follows directly from the relative compounding effects associated with g_i and c_i. Where $G > C$ for all months, the effect of compounding from paying early is greater than that from paying late; *i.e.*, if $G \approx 2C$, $g_i > 2c_i$. The effect of compounding, for c_i as defined in chapter 2, are presented immediately. The effect on the corporation's loss from moving $1,000 from the 12th payment date to the 1st payment date is presented in Table 4.2. Where $G > C$, the corporation can always improve its position through decreasing (to zero) any amount paid after the first payment date. Note that this effect is relatively small.

TABLE 4.2

The Dollar Decrease in the Corporation's Loss from Decreasing the 12th Instalment by $1,000 and Increasing the 1st Instalment by $\$1{,}000 * \frac{g_{12}}{g_1}$

G	C								
	.04	.05	.06	.07	.08	.09	.10	.11	.12
.04	0	-	-	-	-	-	-	-	-
.05	.22	0	-	-	-	-	-	-	-
.06	.44	.28	0	-	-	-	-	-	-
.07	.67	.56	.33	0	-	-	-	-	-
.08	.89	.84	.67	.39	0	-	-	-	-
.09	1.12	1.13	1.02	.79	.45	0	-	-	-
.10	1.35	1.42	1.36	1.20	.91	.51	0	-	-
.11	1.59	1.71	1.71	1.60	1.38	1.04	.58	0	-
.12	1.82	2.00	2.06	2.01	1.85	1.56	1.16	.64	0
.13	2.05	2.29	2.42	2.42	2.32	2.09	1.75	1.29	.70
.14	2.29	2.59	2.77	2.88	2.79	2.63	2.34	1.94	1.42

CHAPTER 5

NUMERICAL OPTIMIZATION

5.1 Introduction

In this chapter, numerical optimization in the form of linear programming is utilized to determine the corporation's optimal contingent payment vector. Section 5.2 formulates a linear program which minimizes the corporation's expected loss set out in equation (4.3) without most of the restrictions imposed on the analytic solutions in chapter 4: that is, without the assumption that interest rates are simple and non-changing, and without the assumption of no stub loss. In section 5.3 a simple quarterly binomial information structure is presented. A discussion of the application of this formulation, and an example of the implementation of numerical optimization, follows in section 5.4. Recall that the expected loss set out in equation (4.3) assumed perfect foresight about interest rates. The final section relaxes this assumption by setting out a formulation of the linear programming problem for stochastic rates.

5.2 A General Linear Programming Formulation

The corporation's optimization problem may be reformulated as minimizing a linear function subject to a finite number of linear constraints, and therefore may be solved using linear programming techniques.[53] The objective function in this problem is to minimize the expected value of the loss as set out in equation (4.3) in chapter 4. In converting the problem to linear programming form, the first step is to replace three of the maximization operators by non-negativity constraints; the operator associated with instalment interest (the first term in equation (4.3)) and from the stub loss (the fourth term in (4.3)), and the first operator in the penalty (the third term in equation (4.3)). For example, the constraints $\overline{U_\omega} \geq \sum_{i=1}^{12}\left(q_i^\omega - p_{ij}\right)g_i$ and $\overline{U_\omega} \geq 0$ are equivalent to $\overline{U_\omega} \geq \max\left(0, \sum_{i=1}^{12}\left(q_i^\omega - p_{ij}\right)g_i\right)$. In the optimum, $\overline{U_\omega} = \max\left(0, \sum_{i=1}^{12}\left(q_i^\omega - p_{ij}\right)g_i\right)$ since $\overline{U_\omega}$ is part of the objective function to be minimized and hence, the linear programming algorithm will eliminate any slack in the inequality.

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[53]This problem could also be solved numerically using non-differentiable optimization methods (for a review of relevant software, see More [1993]). However, non-differentiable optimization software is much less readily available and the computational efficiency is likely to be lower.

The linear programming problem may therefore be written,

$$\textit{minimize} \quad \sum_{\omega=1}^{\Omega} \left(\overline{U_\omega} + \sum_{i=1}^{12} \left(p_{ij} - q_i^\omega\right) c_i + \overline{Pen_\omega} + \overline{Stub_\omega} \right) Prob_\omega$$

subject to

$$\overline{U_\omega} \geq \sum_{i=1}^{12} \left(q_i^\omega - p_{ij}\right) g_i \qquad \forall \ \omega \in \Omega \tag{5.1}$$

$$\overline{Pen_\omega} \geq .50 \cdot \left[\sum_{i=1}^{12} \left(q_i^\omega - p_{ij}\right) g_i \ - \ \max\left(1000, .25 \sum_{i=1}^{12} q_i^\omega g_i\right) \right] \qquad \forall \ \omega \in \Omega$$

$$\overline{Stub_\omega} \geq \sum_{i=1}^{12} \left(p_{ij} - x^\omega\right) s_{yz} \qquad \forall \ \omega \in \Omega$$

with non-negativity constraints: $p_{ij} \geq 0 \ \ \forall \, i,j$; and $\overline{U_\omega}$, $\overline{Pen_\omega}$, $\overline{Stub_\omega} \geq 0 \ \ \forall \, \omega$. In the optimum, $\overline{U_\omega}$ is the underpayment loss in state ω, $\overline{Pen_\omega}$ is the penalty in state ω, and $\overline{Stub_\omega}$ is the stub loss in state ω. The endogenous variables are the vector of contingent payments, and $\overline{U_\omega}$, $\overline{Pen_\omega}$, $\overline{Stub_\omega}$ for each ω. The exogenous variables are: x^ω for all ω; b_1 and b_2 (which, together with x^ω, allow us to fully define q_i^ω for all ω); and C_i and G_i for all i (which allows us to define c_i, g_i, and s_{yz}). Note that this formulation contains a maximization operator in $\overline{Pen_\omega}$. This is not a problem for the linear programming algorithm because, as each of the elements inside the operator is exogenous, $\overline{Pen_\omega}$ is a parameter and may be solved for outside the linear programming formulation (in the same way that q_i^ω may be determined outside the program).

Given the above linear programming formulation, the number of constraints and variables may be calculated as follows. Let us first determine the number of constraints. Suppose that there is one event at date 1 (as assumed previously) and each event at date i is the union of k events at date $i+1$, for all $i = 1,2,...,12$. Hence, there are k^{12} states of nature. As the linear programming formulation requires three constraints for each state of nature, the number of constraints is $3k^{12}$. Let us now determine the number of variables (the number of contingent payments). In month 1 the corporation determines a single contingent payment, in month 2 it calculates k contingent payments, in month 3 it determines k^2 contingent payments, *etc.* The number of endogenous variables is therefore $1 + k + k^2 + ... + k^{11}$ or $\sum_{i=1}^{12} k^{i-1}$. For example, if $k = 3$ (a trinomial information structure), the linear programming problem would have 531,441 constraints and 245,720 variables. Solving a problem of this magnitude would be computer intensive.

5.3 Linear Programming Formulation for a Quarterly Information Model

In this section, the information structure is simplified to reduce the size of the linear programming problem. Let us assume that information about tax liability arrives within one month of the end of each fiscal quarter; that is, at only four times in the instalment period rather than monthly. A justification for this assumption would be that corporations will update instalment payments only on receiving information in the form of four quarterly financial statements (either internal or external reports). The effect of this assumption is to make the partitions of Ω identical within a quarter: $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3$; $\mathcal{F}_4 = \mathcal{F}_5 = \mathcal{F}_6$; $\mathcal{F}_7 = \mathcal{F}_8 = \mathcal{F}_9$; and

$\mathcal{F}_{10} = \mathcal{F}_{11} = \mathcal{F}_{12}$. Given this quarterly information structure, the number of states of nature is reduced from k^{12} to k^4. The number of constraints is therefore reduced from $3k^{12}$ to $3k^4$. The number of variables can be determined as follows. Since information is received only once per quarter, the number of variables for each month in a particular quarter is the same. Therefore, the number of variables is $1+1+1+k+k+k+k^2+k^2+k^2+k^3+k^3+k^3$ or $3\cdot\sum_{i=1}^{4} k^{i-1}$. In moving from monthly information to quarterly information, the number of variables is therefore reduced from $\sum_{i=1}^{12} k^{i-1}$ to $3\cdot\sum_{i=1}^{4} k^{i-1}$.

The choice of k is an important determinant in the size of the problem. Let us choose $k=2$ (a binomial information structure) to further restrict the size of the quarterly information structure. This formulation therefore has 48 constraints $(3\cdot 2^4)$ and 45 variables $\left(3\cdot\sum_{i=1}^{4} 2^{i-1}\right)$.

In this quarterly information model, any state of nature belongs to a unique set of five events (of which the first event is common for all states of nature). Therefore, let a state of nature, ω, which corresponds to a particular series of 4 events be denoted by a four letter combination of *u* ("upticks" or "good news") and *d* ("downticks" or "bad news"); *i.e.*, *uddu* is the state represented by an uptick in the first quarter, a downtick in the second and third quarters, and an uptick in the fourth quarter. Any particular event is denoted by $\phi_{z,j}$ where

z is the quarter and j is the history at that date. For example, consider the event $\phi_{3,ud}$={*uduu,udud,uddu,uddd*}; that is, in the event in quarter 3 with history *ud*, the corporation knows that the state of nature is one of *uduu*, *udud*, *uddu*, or *uddd*. Table 5.1 sets out the fifteen events.[54]

[54]To simplify exposition, the events at the reminder due date are not presented.

TABLE 5.1

Quarterly Information Structure F = $\{\mathcal{F}_{Q1}, \mathcal{F}_{Q2}, \mathcal{F}_{Q3}, \mathcal{F}_{Q4}\}$ for Events $\phi_{z,j}$ in Quarter z

$\mathcal{F}_{Q1} = \Omega$

$\mathcal{F}_{Q2} = \{\phi_{1,u}, \phi_{1,d}\}$

$\mathcal{F}_{Q3} = \{\phi_{2,uu}, \phi_{2,ud}, \phi_{2,du}, \phi_{2,dd}\}$

$\mathcal{F}_{Q4} = \{\phi_{3,uuu}, \phi_{3,uud}, \phi_{3,udu}, \phi_{3,udd}, \phi_{3,duu}, \phi_{3,dud}, \phi_{3,ddu}, \phi_{3,ddd}\}$

where,

$\phi_1 = \{uuuu, uuud, uudu, uudd, uduu, udud, uddu, uddd, duuu, duud, dudu, dudd, dduu, ddud, dddu, dddd\}$

$\phi_{2,u} = \{uuuu, uuud, uudu, uudd, uduu, udud, uddu, uddd\}$

$\phi_{2,d} = \{duuu, duud, dudu, dudd, dduu, ddud, dddu, dddd\}$

$\phi_{3,uu} = \{uuuu, uuud, uudu, uudd\}$

$\phi_{3,ud} = \{uduu, udud, uddu, uddd\}$

$\phi_{3,du} = \{duuu, duud, dudu, dudd\}$

$\phi_{3,dd} = \{dduu, ddud, dddu, dddd\}$

$\phi_{4,uuu} = \{uuuu, uuud\}$

$\phi_{4,uud} = \{uudu, uudd\}$

$\phi_{4,udu} = \{uduu, udud\}$

$\phi_{4,udd} = \{uddu, uddd\}$

$\phi_{4,duu} = \{duuu, duud\}$

$\phi_{4,dud} = \{dudu, dudd\}$

$\phi_{4,ddu} = \{dduu, ddud\}$

$\phi_{4,ddd} = \{dddu, dddd\}$

It is therefore evident, given this defined information structure, that events become finer with the release of additional information: that is, for any quarter $k > i$, $\phi_{k,j} \subset \phi_{i,j}$. For example, from Table 5.1 we see that $\phi_{4,ddu} \subset \phi_{3,dd} \subset \phi_{2,d} \subset \phi_1$.

Diagrammatically, this information structure may be represented as in Figure 5.1. From Figure 5.1, we see that the first three monthly payments (January, February, and March), are made based on the event ϕ_1 which equals Ω; that is, the corporation will make these payments knowing that the terminal state will take one of 16 values (16 terminal states ω) each with some corresponding probability $Prob_\omega$. In the first month of the second quarter, the corporation receives news, either good (*u*) or bad (*d*), from the first quarter. Therefore the information known to the corporation at the time it makes its April, May, and June payments is represented by either the event $\phi_{2,u}$ (the information at quarter 2 with history *u*) or $\phi_{2,d}$ (the information at quarter 2 with history *d*). At information date 3 the event will reflect the additional good news or bad news in the third quarter such that the information history is either *uu*, *ud*, *du*, or *dd*, with events $\phi_{3,uu}$, $\phi_{3,ud}$, $\phi_{3,du}$, or $\phi_{3,dd}$. The corporation will make its July, August, and September payments based on that event. Similarly, at information date 4, the good news or bad news from the third quarter, in addition to the information from prior quarters, will be used in deciding the amount to pay in October, November, and December with some history *uuu*, *uud*, *udu*, *udd*, *duu*, *dud*, *ddu*, or *ddd* (with events $\phi_{4,uuu}$, $\phi_{4,uud}$, $\phi_{4,udu}$, $\phi_{4,udd}$, $\phi_{4,duu}$, $\phi_{4,dud}$, $\phi_{4,ddu}$, and $\phi_{4,ddd}$ respectively).

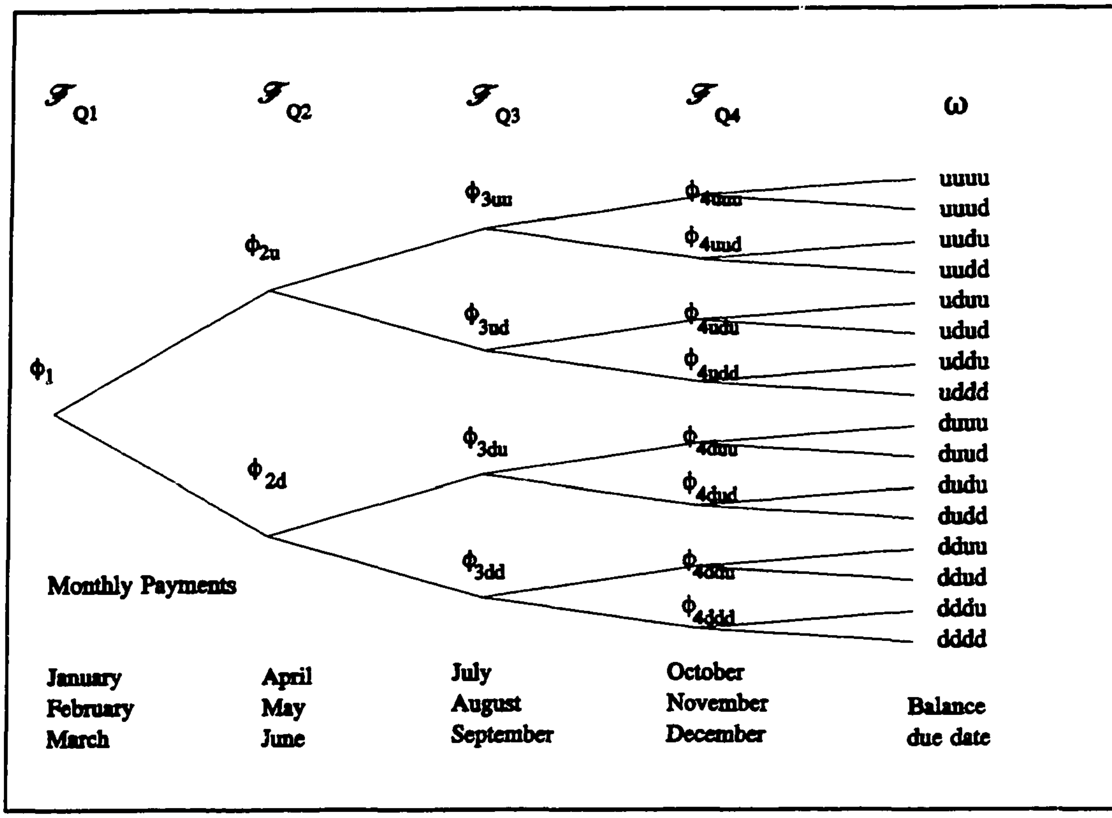

FIGURE 5.1

Information and Payment Structure

Corresponding to each event ϕ_{tj} are three contingent payments p_{ij}. For January, February and March, there is only one event (ϕ_1), so there are just the three payments to be determined: p_1, p_2, and p_3. For April, May and June, there are two events ("uptick" and "downtick"). Therefore, two contingent payments must be determined for each of these three months for a total of 6: p_{4u}, p_{4d}, p_{5u} p_{5d}, p_{6u}, and p_{6d}. Similarly, as there are four events in the third quarter and eight events in the fourth quarter, the corporation must determine four contingent payments for each month in the third quarter and eight contingent payments for each month in the fourth quarter. The corporation will therefore optimally determine the 45 contingent payments set out in Table 5.2.

TABLE 5.2

Contingent Payments			
Jan., Feb., March	April, May, June	July, Aug., Sept.	Oct., Nov., Dec.
p_1, p_2, p_3	p_{4u}, p_{5u}, p_{6u}	$p_{7uu}, p_{8uu}, p_{9uu}$	$p_{10uuu}, p_{11uuu}, p_{12uuu}$
⋮	⋮	⋮	$p_{10uud}, p_{11uud}, p_{12uud}$
⋮	⋮	$p_{7ud}, p_{8ud}, p_{9ud}$	$p_{10udu}, p_{11udu}, p_{12udu}$
⋮	⋮	⋮	$p_{10udd}, p_{11udd}, p_{12udd}$
⋮	p_{4d}, p_{5d}, p_{6d}	$p_{7du}, p_{8du}, p_{9du}$	$p_{10duu}, p_{11duu}, p_{12duu}$
⋮	⋮	⋮	$p_{10dud}, p_{11dud}, p_{12dud}$
⋮	⋮	$p_{7dd}, p_{8dd}, p_{9dd}$	$p_{10ddu}, p_{11ddu}, p_{12ddu}$
⋮	⋮	⋮	$p_{10ddd}, p_{11ddd}, p_{12ddd}$

The linear programming formulation for quarterly information, specializing equation (5.1) to this quarterly information structure, is,

$$\begin{aligned} \textit{minimize} \quad & \left(\overline{U_1} + \sum_{i=1}^{12}\left(p_{ij}-q_i^1\right)c_i + \overline{Pen_1} + \overline{Stub_1}\right) Prob_1 \\ & + \left(\overline{U_2} + \sum_{i=1}^{12}\left(p_{ij}-q_i^2\right)c_i + \overline{Pen_2} + \overline{Stub_2}\right) Prob_2 \\ & + \qquad \cdots \\ & + \left(\overline{U_{16}} + \sum_{i=1}^{12}\left(p_{ij}-q_i^{16}\right)c_i + \overline{Pen_{16}} + \overline{Stub_{16}}\right) Prob_{16} \end{aligned} \tag{5.2}$$

subject to

$$\begin{aligned} \overline{U_1} &\geq \sum_{i=1}^{12}\left(q_i^1 - p_{ij}\right)g_i \\ &\vdots \\ \overline{U_{16}} &\geq \sum_{i=1}^{12}\left(q_i^{16} - p_{ij}\right)g_i \\ \overline{Pen_1} &\geq \sum_{i=1}^{12}\left(q_i^1 - p_{ij}\right)g_i - \max\left(1000, .25\sum_{i=1}^{12} q_i^1 g_i\right) \\ &\vdots \\ \overline{Pen_{16}} &\geq \sum_{i=1}^{12}\left(q_i^{16} - p_{ij}\right)g_i - \max\left(1000, .25\sum_{i=1}^{12} q_i^{16} g_i\right) \\ \overline{Stub_1} &\geq \sum_{i=1}^{12}\left(p_{ij} - x^1\right)s_{yz} \\ &\vdots \\ \overline{Stub_{16}} &\geq \sum_{i=1}^{12}\left(p_{ij} - x^{16}\right)s_{yz} \end{aligned} \tag{5.3}$$

where c_i, g_i, s_{yz}, and q_i^{ω} are defined in equations (2.14), (2.10), (2.22), and (4.4) respectively.

5.4 Practical Implementation Strategy for the Model

Using an assumption of deterministic rates, but allowing updating for mistakes, a potential implementation strategy for corporations which could be used by tax practitioners is set out. Under this strategy, a corporation pays as if rates are deterministic, but knowing that they are not, resolves the problem as rates change. Note that this method has little theoretical purity, but has implementation advantages. An example is then provided to illustrate this strategy.

Let us assume that the corporation's best estimate of its cost of capital in the following months is the current month's cost of capital (that in month i, its best estimate of its cost of capital for months $i+1$ to the remainder due date is the rate at date i). A possible application of the linear programming formulation set out in equation (5.7) would be to make a payment at date 1 as if C_i were constant and G_i followed the a lag process set out below. At date 2, if the rates had changed, the linear programming problem would be solved for all remaining contingent payments using these new rates. If the rate had not changed, the corporation would continue to make the payments from the first optimization. This process would be repeated until date 12. Note that by following this process, the stochastic process is not explicitly modelled, but the corporation reacts to changes in rate structure.

Prior to setting out an example, let us briefly examine the effect of the lag on the prescribed rate. The prescribed rate or rate on underpayment in month i, G_i, is defined in Regulation 4301 for a quarter as the three month Canada Treasury Bill rate (hereafter the T-Bill rate) for the first month of the immediately preceding quarter rounded to the next highest whole percentage plus 2%. That is,

$$G_i = \lfloor T_{i-h} \rfloor + .02 \quad \text{where} \quad \begin{array}{l} h = 4 \text{ for } i = 1,4,7,10 \\ h = 5 \text{ for } i = 2,5,8,11 \\ h = 6 \text{ for } i = 3,6,9,12 \end{array} \tag{5.4}$$

where h is the lag and T_{i-h} is the T-Bill rate in month i-h. Note that the rate for January, February and March is based on the October T-Bill rate for the preceding year. Table 5.3 demonstrates the effect of the lag on the calculation of G_i and the prescribed rates for 1993.

TABLE 5.3

Determining the Rates on Underpayment, G_i

	First Month of Each Quarter for 1993				
	Oct. 1992	Jan. 1993	Apr. 1993	July 1993	Oct. 1993
3 Month T-Bill Rates[55] for First Month in Each Quarter	T_{-3} 7.16%	T_1 6.71%	T_4 5.19%	T_7 4.31%	T_{10} 4.56%
The Rate G_i for Each Month i	G_i for i=1,2,3 10%	G_i for i=4,5,6 9%	G_i for i=7,8,9 8%	G_i for i=10,11,12 7%	G_i for i=13,14,15 7%

[55]The T-Bill rates are from the Bank of Canada Review [1994].

Let us assume that a Canadian corporation, Ray Co., provides the following information:

Tax liability for 1993 (1st instalment base)	$12,000,000
Tax liability for 1992 (2nd instalment base)	$9,600,000
Tax liability for 1994 where Ray Co. has:	
4 "good" quarters	$14,400,000
3 "good" quarters	$10,800,000
2 "good" quarters	$7,200,000
1 "good" quarters	$3,600,000
0 "good" quarters	0
Date of refund (if any)	120 days after year end

Let us assume that in a good quarter, Ray Co. has tax liability of $1,200,000 per month and in a bad quarter a tax liability of zero (which is consistent with the above quarterly structure). The corporation further assesses the probability of a good quarter to be equal to the probability of a bad quarter, and therefore the relevant probabilities are: 4 good quarters, 1/16; 3 good quarters, 1/4; two good quarters, 3/8; 1 good quarter, 1/4; and zero good quarters, 1/16.

For G_i, let us use the prescribed rates set out in Table 5.3 above. We require an additional assumption with respect to the corporation's after-tax cost of capital. Let us assume that its before tax cost of capital in any month i is 3% above the 3 month T-Bill rate in month i (where the T-Bill rate is rounded to the next highest percent).[56] Therefore, Ray Co.'s after-tax cost of capital for month i is $(.03 + \textit{T-Bill rate}_i) \cdot (1 - \tau)$, where τ is its marginal tax rate.

[56]This assumption is made solely for convenience. Any mapping of the corporation's cost of capital into the T-Bill rate could be utilized.

Let us assume a rate $\tau = 40\%$. Table 5.4 sets out the corporation's expectation as to the rates C_i and G_i for that month i and all months following i.

TABLE 5.4

Expected Rates C_i and G_i at Each Payment Date													
		Months											
		Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
T-Bill Rounded		7	7	6	6	6	5	5	5	5	5	5	4
Expected Rates C_i, G_i at date i	1	5.4,10	5.4,10	5.4,10	5.4,9	5.4,9	5.4,9	5.4,9	5.4,9	5.4,9	5.4,9	5.4,9	5.4,9
	2		5.4,10	5.4,10	5.4,9	5.4,9	5.4,9	5.4,9	5.4,9	5.4,9	5.4,9	5.4,9	5.4,9
	3			4.8,10	4.8,9	4.8,9	4.8,9	4.8,8	4.8,8	4.8,8	4.8,8	4.8,8	4.8,8
	4				4.8,9	4.8,9	4.8,9	4.8,8	4.8,8	4.8,8	4.8,8	4.8,8	4.8,8
	5					4.8,9	4.8,9	4.8,8	4.8,8	4.8,8	4.8,8	4.8,8	4.8,8
	6						4.2,9	4.2,8	4.2,8	4.2,8	4.2,7	4.2,7	4.2,7
	7							4.2,8	4.2,8	4.2,8	4.2,7	4.2,7	4.2,7
	8								4.2,8	4.2,8	4.2,7	4.2,7	4.2,7
	9									4.2,8	4.2,7	4.2,7	4.2,7
	10										4.2,7	4.2,7	4.2,7
	11											4.2,7	4.2,7
	12												3.6,7

To demonstrate how these expected rates were determined, let us examine Ray Co.'s expectation as to rates in June (at payment date 6). In June, Ray Co.'s after-tax cost of capital is 4.2% $[(.05+.02)\cdot(1-.4) = .042]$. It will therefore determine contingent payments, given the "fixed" past payments, based on a future cost of capital (for June through December and the stub period) based on a cost of capital equal to 4.2%. In June, Ray Co. knows with certainty the rates G_i for June, July, August, and September, and will use the T-Bill rate in June to form expectations about the rate in October, November, December, and the stub period. The June rate, G_6, equals the January T-Bill rate, rounded to the next highest full percentage, plus 2%, or 9%. The July, August, and September rates are based on the April T-Bill rate and equal 8%. For all months after September (including the stub period), Ray Co.'s expectation is that G_i is 7% (as the June T-Bill rate equals 5%). Note that for expositional purposes, Table 5.4 does not include the expected rates in the stub period, although, as noted above, they are determined in the same manner as the other rates.

Using the updating procedure described above, Ray Co., updating its payment strategy monthly, would make the following contingent payments:[57]

January	Quarter 1 Outcome	April	Quarter 2 Outcome	July	Quarter 3 Outcome	October
2,013,580	down	0	down	0	down	0
					up	3,453,806
			up	1,822,307	down	0
					up	5,707,968
	up	2,094,181	down	0	down	0
					up	3,037,765
			up	2,370,471	down	0
					up	4,321,768

Let us first note that for any combination of good and bad quarters, the corporation will only make payments in the first month or each quarter; as the optimal payment amount in February, March, May, June, August, September, November, and December is zero, they are not presented above.

The expected loss associated with these updated contingent payments is $42,773. It is worthwhile comparing this expected loss with certain other payment strategies. First, let us assume that the corporation has perfect foresight with respect to rates (a very extreme assumption). The expected loss under that assumption is $42,202; a decrease in expected loss of less than 1% (other examples tended to provide similarly small decreases in expected loss). The implication of this result, which is admittedly based on a single hypothetical example, is

[57]For all the examples in this chapter and in chapter 6 the definition of instalment liability used is that which applied to fiscal years prior to 1992. This should not have a significant effect on the results.

that the modelling of stochastic rates may be, in a practical sense, unimportant. That is, a stochastic model must, by definition, have an expected loss greater than the perfect foresight model. As the expected loss from following the updating process described above is only marginally greater than the expected loss from the perfect foresight model, it would tentatively appear that the gain from incorporating the stochastic process into the linear programming formulation might not exceed the costs of working with a much larger model (the size of the stochastic model is developed in the next section).

Let us now compare the expected loss where the corporation resolves the problem as rates change, to where the corporation does not update its contingent payments for changes in rates. The expected loss where the corporation does not update rates is $52,915; an increase in expected loss in excess of 20% over the expected loss with updating. This implies that while it may not be worthwhile to model stochastic rates, it is beneficial to update payment strategies to reflect changes in rates.

5.5 Changing Rates of Interest

The expected loss developed in equation (4.3) of chapter 4 modeled the corporation's tax liability for the year as stochastic. It was assumed in that formulation, and therefore in the linear programming formulation in section 5.1, that interest rates were deterministic. In this section that assumption will be relaxed and a linear programming formulation with stochastic rates is presented. As is demonstrated in this section, relaxing the assumption of deterministic rates is, analytically, a straightforward extension. However, stochastic rates greatly increase the size of the problem making it difficult to solve.

5.5.1 The Expected Loss

An information structure where rates are stochastic may take a similar form to that presented in chapter 4. Let us assume that both the corporation's tax liability for the year and rates are stochastic. Therefore, for each state of nature, ω, there is a path of associated rates, G_i and C_i, as well as an associated value of the monthly-average tax liability for the year, x. For example, if the corporation's monthly-average tax liability is binomial (either "up" or "down"), and the rate structure is similarly binomial (either "increase" or "decrease"), then each event at date i is the union of 4 events at date $i+1$ (the number of combinations of "up" or "down" with "increase" or "decrease"). In other words, this is a 4-nomial information structure.

The expected loss function where both the corporation's tax liability for the year and rates are stochastic is therefore,

$$L(p) = \sum_{\omega=1}^{\Omega} \Bigg\{ \max\left[0, \sum_{i=1}^{12} (q_i^\omega - p_{ij}) g_i^\omega\right] + \sum_{i=1}^{12} (p_{ij} - q_i^\omega) c_i^\omega$$

$$+ .50 * \max\left[0, \sum_{i=1}^{12} (q_i^\omega - p_{ij}) g_i^\omega - \max\left(1000, .25 \sum_{i=1}^{12} q_i^\omega g_i^\omega\right)\right] \quad (5.5)$$

$$+ \max\left[0, \sum_{i=1}^{12} \left(p_{ij} - x^\omega\right) s_{yz}^\omega\right] \Bigg\} Prob_\omega$$

where

$$q_i^\omega = \begin{cases} x^\omega & \forall\ i = 1 \text{ to } 12 & \text{if } x^\omega \le b_1 \\ b_1 & \forall\ i = 1 \text{ to } 12 & \text{if } b_1 \le \{x^\omega, b_2\} \\ \begin{cases} b_2 & \forall\ i = 1,2 \\ \frac{1}{10}(12 b_1 - 2 b_2) & \forall\ i = 3 \text{ to } 12 \end{cases} & & \text{if } b_2 \le b_1 \le x^\omega \end{cases} \quad (5.6)$$

where c_i^ω, g_i^ω, and s_{yz}^ω are determined by the stochastic rates C_i^ω and G_i^ω. Note that this formulation is identical to the expected loss set out in equation (4.3) except that the rates are now a function of the state of nature.

5.5.2 Linear Programming Formulation for Stochastic Rates of Interest

The linear programming formulation of the expected loss set out in equation (5.5) is similar to the formulation in equation (5.1) where rates are not stochastic. The expected loss (noting that the states ω now reflect both a rate history and tax liability history) is

$$\text{minimize} \quad \sum_{\omega\in\Omega}\left(\overline{U_\omega} + \sum_{i=1}^{12}\left(p_{ij}-q_i^\omega\right)c_i^\omega + \overline{Pen_\omega} + \overline{Stub_\omega}\right)Prob_\omega$$

subject to

$$\overline{U_\omega} \geq \sum_{i=1}^{12}\left(q_i^\omega - p_{ij}\right)g_i^\omega \quad \forall \ \omega \in \Omega \tag{5.7}$$

$$\overline{Pen_\omega} \geq \sum_{i=1}^{12}\left(q_i^\omega - p_{ij}\right)g_i^\omega - \max\left(1000, .25\sum_{i=1}^{12} q_i^\omega g_i^\omega\right) \quad \forall \ \omega \in \Omega$$

$$\overline{Stub_\omega} \geq \sum_{i=1}^{12}\left(p_{ij} - x^\omega\right)s_{yz}^\omega \quad \forall \ \omega \in \Omega$$

The calculation of constraints and variables for this model is the same as that set out in section 5.2 above. Note however that as the number of branches from each event are increased (*k* is larger), the number of constraints and variables increases significantly. For example, where binomial information relating to tax liability for the year arrives each month, but rates are not stochastic, the number of constraints is $3 \cdot 2^{12} = 12{,}288$ and the number of variables is $\sum_{i=1}^{12} 2^{i-1} = 4{,}195$. In adding a binomial information structure for rates (assuming that C_i and G_i are mechanically related), there are four branches from each event ($k = 4$), the number of constraints is $3 \cdot 4^{12} = 16{,}768{,}461$ and the number of variables is $\sum_{i=1}^{12} 4^{i-1} = 16{,}777{,}216$. It is therefore evident that one would need to greatly restrict the number of information points for it to be feasible to solve this linear program. For example, with a quarterly information structure, the problem would have 768 constraints and 341 variables.

5.5.3 Relation to Interest Rate Literature

The above formulation of stochastic interest rates assumes that information is binomial and arrives quarterly. These assumptions are made to illustrate the effect that stochastic interest rates have on the size of the problem. Much more severe size difficulties would arise if information arrived more frequently or if the structure were n-nomial instead of binomial, as models of the structure of interest rates usually assume (Vetzal [1994]).

The above formulation also does not discuss the practical issues involved in determining interest rate forecasts and associated probabilities for a particular taxpayer. Such issues include the dependence of interest rates on macroeconomic variables and the tendency of interest rates to display mean reversion (Vetzal [1994]).

CHAPTER 6

POLICY IMPLICATIONS

6.1 Introduction

The effects of instalment rules can be summarized in the form of rate effects. This chapter demonstrates this point by developing measures which may be interpreted as average and marginal effective tax rates under the instalment structure.

There exists an extensive body of literature in accounting and economics which is concerned with the determination of effective tax rates and their application.[58] Boadway [1985, 63] notes the applicability of these measures to policy formulation,

> [s]ince they [*effective tax rates*] are a measure of the size of the tax distortion in various lines of activity, they serve as a useful guide to policy makers who are interested in knowing, for tax reform purposes, which activities are currently favoured and which are discriminated against, and by how much.

Messere [1993, 332] recognizes that an instalment structure will have effective tax rate effects,

> [t]hese different methods (*of tax collection*) do affect ... effective rates of tax and, accordingly, could influence government decisions on statutory rates and coverage.

This chapter is structured as follows. In section 6.2, a method is developed to determine the corporation's average and marginal effective tax rate. To demonstrate potential uses of the resulting measures in determining policy, an example is provided in section 6.3.

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[58]This literature, starting with Hall and Jorgenson [1967], is summarized by Callihan [1992].

Three things are examined in this example: first, the effect on the corporation of a change in instalment structure; second, the effect on the corporation of paying suboptimally; and third, the effect on the corporation of changing parameter values. Note that the results are provided to demonstrate potential policy uses for the measures; the associated numbers are not in themselves significant. However, the relative magnitudes do provide some indication of the relative importance of different potential amendments.

6.2 Developing the Methods

To determine the effect of the instalment structure on the corporation's tax liability, let us proceed as follows. First, let us calculate the expected present value as of the first payment date of optimal payments under the instalment structure; that is, as of the first payment date. Second, let us calculate the expected present value of instalment payments under an instalment structure which requires that tax be paid as income is earned; the expected present value of the benchmark. These amounts will then be used to create measures which may be viewed as average and marginal tax rates.

6.2.1 Calculating the Corporation's Present Value of Payments

Let us calculate the expected value of all payments required by the Income Tax Act to be paid to Revenue Canada in respect of the year. As was demonstrated in Appendix C, the present value of all payments is a linear transformation of $l(p;x)$;

$$t_{PV} = a_1 + a_2\, l(p;x) \tag{6.1}$$

where,

$$a_1 = \frac{\sum_{i=1}^{12} q_i c_i + \sum_{i=1}^{12} x}{\prod_{j=0}^{12} \left(1 + \frac{C}{365}\right)^{N_{j,j+1}}}$$

and,

$$a_2 = \frac{1}{\prod_{j=0}^{12} \left(1 + \frac{C}{365}\right)^{N_{j,j+1}}}$$

Recall that, from this relationship, minimizing the expected value of $l(p;x)$ is equivalent to minimizing the expected value of $t_{pv}(p;x)$; *i.e.*, both problems produce the same optimal values of the decision variables, p.

Taking the expectation of equation (6.1), one may determine the expected present value of all payments;

$$E[t_{PV}] = E[a_1(\omega)] + a_2\, E[l(p;x^{\omega})] \tag{6.4}$$

Note that $E[l(p;x^{\omega})]$ is the objective function set out in equation (5.1) in chapter 5. As

$E[a_1(\omega)]$ is exogenous (although it is a function of x, it is not a function of p), it may be determined outside the optimization problem (*i.e.*, outside the linear programming formulation).

6.2.2 Calculating a Benchmark Amount

It is useful to compare the expected present value of payments determined above against a benchmark or theoretical standard; that is, against a constructed liability structure which has as economic interpretation. This benchmark is that corporations pay tax liability month by month as income is earned. This may be justified as an appropriate benchmark on both normative and positive grounds. One normative justification is that this measure is consistent with the Haig-Simons concept of measuring income on an accrual basis. A second normative justification is that the measure is consistent with how Canadian wage earners pay tax: that is, it is consistent with how income tax is withheld from earnings. A positive argument for the use of this benchmark is that the effective tax rate literature has assumed that tax liability is paid as income is earned and therefore the effective tax rates developed here can be used in place of the statutory rate in the formulas for effective tax rates developed in papers such as Mackenzie [1994] and Daly *et.al.* [1993]. Through such a procedure, it would be possible to develop a more comprehensive tax rate measure which incorporates the effects of both the instalment structure and the capital cost allowance, investment tax credits, *etc.* studied by these other authors.

The benchmark amount is therefore the sum of the discounted liabilities. To simplify analysis, it is assumed that income is earned in discrete intervals, in each month, and that this associated liability is discounted to January 31. As the tax liability for the year is uncertain, an expected discounted liability must be calculated. This may be written, for any monthly tax liability x_i, as

$$\begin{aligned} E(x_{pv}) &= E\left[\sum_{i=1}^{12} \frac{x_i^{\omega}}{\prod_{j=0}^{i-1}\left(1+\frac{C}{365}\right)^{N_{j,j+1}}}\right] \\ &= \sum_{i=1}^{12} \frac{E\left(x_i^{\omega}\right)}{\prod_{j=0}^{i-1}\left(1+\frac{C}{365}\right)^{N_{j,j+1}}} \end{aligned} \tag{6.5}$$

6.2.3 Calculating the Effect on Effective Tax Rates of the Instalment Structure

The corporation's average effective tax rate (hereafter AETR) is defined as,

$$AETR = \left(\frac{E(t_{pv})}{E(x_{pv})}\right) \cdot s \tag{6.6}$$

The corporation's marginal effective tax rate (hereafter METR) may be defined as,

$$METR = \left(\frac{\Delta E(t_{pv})}{\Delta E(x_{pv})}\right) \cdot s \tag{6.7}$$

where $\Delta E(t_{pv})$ and $\Delta E(x_{pv})$ represent the change in $E(t_{pv})$ and $E(x_{pv})$ respectively for a 1%

increase in the corporation's tax liability for the year in each state of nature.[59]

6.3 Examples Using the AETR and METR Measures

To demonstrate the application of the above measures, let us first examine the rate effects on a single corporation for a given set of parameters; the Ray Co. example from chapter 5. Assume that Ray Co. provides the following information:

Tax liability for 1993 (1st instalment base)	$12,000,000
Tax liability for 1992 (2nd instalment base)	$9,600,000
Tax liability for 1994 where Ray Co. has:	
4 "good" quarters	$14,400,000
3 "good" quarters	$10,800,000
2 "good" quarters	$7,200,000
1 "good" quarters	$3,600,000
0 "good" quarters	0
Date of refund (if any)	120 days after year end

Let us assume that in a good quarter, Ray Co. has tax liability of $1,200,000 per month and in a bad quarter a tax liability of zero (which is consistent with the above quarterly structure). The corporation further assesses the probability of a good quarter to be equal to the probability of a bad quarter, and therefore the relevant probabilities are: 4 good quarters, 1/16; 3 good

[59]An alternative measure would be to add $1 to tax liability for the year in each state of nature. This would be consistent with Shevlin [1990], but it does not seem as realistic as the multiplicative assumption used here.

quarters, 1/4; two good quarters, 3/8; 1 good quarter, 1/4; and zero good quarters, 1/16. Let us further assume that Ray Co.'s after tax cost of capital is 7% $(C_i = 7\% \ \forall i)$, and the prescribed rate is 11% $(G_i = 11\% \ \forall i)$, from the first payment date to the refund date.

Ray Co's expected value of all payments under the instalment structure is,

$$E[t_{pv}] = 6{,}968{,}465 + (.927407 \cdot 65{,}935) = \$7{,}029{,}613$$

as

$$E(a_1) = \frac{\sum_{i=1}^{12} E(q_i c_i) + E(X)}{\prod_{j=0}^{12} \left(1 + \frac{C}{365}\right)^{N_{j,j+1}}} = \$6{,}968{,}465$$

$$a_2 = \frac{1}{\prod_{j=0}^{12} \left(1 + \frac{C}{365}\right)^{N_{j,j+1}}} = .927407$$

and, given that the Ray Co. pays optimally,

$$E(l(p;x)) = \$65{,}935$$

The benchmark present value is,

$$E(x_{pv}) = \sum_{i=1}^{12} \frac{E(x_i)}{\prod_{j=0}^{12} \left(1 + \frac{C}{365}\right)^{N_{j,j+1}}} = \$6{,}976{,}055$$

Ray Co's AETR, from equation (6.6) is therefore,

$$AETR = \left(\frac{E(t_{pv})}{E(x_{pv})}\right) \cdot s = \frac{7{,}029{,}613}{6{,}976{,}055} \cdot .443 = .4464 \tag{6.8}$$

The rate s used in the examples in this section is the statutory combined Federal-Ontario corporate rate for 1994.[60] Thus, for this taxpayer, the actual instalment structure is less favourable than the benchmark structure. Note that a .001 change in s (*eg*. 44.3% to 44.4%), has a revenue effect of $42 million per year to the government.[61] Hence, a .001 change in AETR would have approximately the same $42 million affect on the present value of tax revenues.

Let us now calculate Ray Co.'s METR. Increasing Ray Co.'s tax liability for the year in every state of nature by 1%, the updated values for the corporation's expected present value of payments is $E(t_{pv}) = \$7{,}099{,}755$ and for the benchmark is $E(x_{pv}) = \$7{,}045{,}815$. Therefore, from equation (6.7),

$$METR = \left(\frac{\Delta E(t_{pv})}{\Delta E(x_{pv})}\right) \cdot s = \frac{7{,}099{,}755 - 7{,}029{,}613}{7{,}045{,}815 - 6{,}976{,}055} \cdot .443 = .4422 \tag{6.9}$$

Note that although both $E(t_{pv})$ and $E(x_{pv})$ increased for a 1% increase in Ray Co.'s tax liability for the year (in every state of nature), $E(t_{pv})$ increased by a smaller amount than $E(x_{pv})$ and therefore Ray Co's METR is less than the statutory rate (is more favourable to the corporation

[60]Note that as the Ontario instalment rules differ from the Federal rules, there may be a bias in using the combined rates (although it is unlikely any bias would be significant).

[61]See Department of Finance [1994,42]

than paying the benchmark amounts).[62]

6.3.1 The Effect on the Corporation of a Change in Instalment Structure

To further demonstrate policy applications of the AETR and METR measures, let us vary the instalment structure (assuming that Ray Co. will vary its payments such that it continues to minimize its expected loss). Table 6.1 summarizes the effects on Ray Co. of varying the instalment structure.

[62]Note that if the corporation knew its instalment liability for the year, then its *AETR* < .443. This would occur as the corporation's instalment liability each month is less than or equal to its tax liability in that month. Note further that the corporation's METR is less than the AETR. This result, as seen in Table 6.3, arises from the differential impact of previous years' tax liability, b_1 and b_2, on the AETR and METR. Where $b_1 = 0$, the corporation's AETR and METR will be equal.

TABLE 6.1

Ray Co's AETR and METR and Percentage Changes in these Rates from the Statutory Rate (44.3%) For Different Instalment Structures

Instalment Structure	AETR	%Δ AETR	METR	%Δ METR
1. Current Canadian instalment structure	44.64	.77	44.22	-.17
2. Rate Structure Changes				
a. Lower G_i by 2%	44.56	.59	44.21	-.21
b. Raise G_i by 3%	44.71	.97	44.24	-.13
c. Make *U* and *Pen* deductible	44.50	.43	44.10	-.45
d. both a. and c.	44.31	.02	43.95	-.80
3. Instalment Liability Changes				
a. Based only on Preceding Years	45.60	2.93	42.36	-4.38
- if b_1 reduced to $4,800,000	43.35	-2.14	42.40	-4.28
b. Based only on Tax Liability for the Year	44.72	.96	44.31	.02
4. Refunds				
a. at the remainder due date	44.61	.71	44.21	-.21
b. delay until date at which interest begins to accrue to corp.	44.66	.80	44.23	-.15
5. Abolish Penalty	44.60	.68	44.40	.23
6. Eliminate Instalments	42.40	-4.28	42.40	-4.28

Let us first examine the effects of changing the rate structure on underpayments; *i.e.*, altering the calculation of G_i. Recall that G_i is calculated as a lagged three month T-Bill rate plus 2 percentage points. Let us examine the effect of increasing and decreasing this rate through changing the number of percentage points added to the T-Bill rate. Further recall that payments to the government of instalment interest and penalty are not deductible. As interest is generally deductible on borrowed amounts, an asymmetry exists. Let us therefore also examine the effect of allowing deductibility.

A question discussed in the professional and academic literature is, should the government "loan" money to corporations at a rate less than, equal to, or greater than the corporation's cost of capital? Justifications have been provided for each of the three alternatives. Stark [1991, 1416] states,

> [I]f interest is intended to eliminate the bias toward borrowing from the government, it should reflect the higher rate at which a taxpayer could otherwise borrow. If intended merely to make the government whole by reimbursing it for its costs of borrowing elsewhere, the rate should reflect the lower rate at which the government can borrow. If the rate is intended to dissuade the taxpayer from borrowing from the government, it should exceed the taxpayer's cost of capital.

The focus of such statements has been on the interest rate charged in isolation of other factors in the instalment structure. From Table 6.1, if the government set G_i equal to the T-Bill rate (that is, at a rate equal to the government's cost of capital), Ray Co.'s AETR and METR would be smaller. Conversely, if the government were to increase the rate it charges (*i.e.*, to G_i plus 3%), Ray Co.'s AETR and METR would be larger. Note however that the changes in Ray Co.'s AETR and METR from moving to either of these structures would be small; each change is less that 1/10th of 1% in the corporation's effective tax rates. Permitting the deduction of interest to the government would, as one would predict, lower the corporation's AETR and METR; if in addition, the government set G_i equal to the T-Bill rate, then rates declined even further. Note that in this latter case the corporation's AETR is almost equal to the statutory rate, although the corporation's METR is .8% less than the statutory rate.

What would be the effect of a fundamental change in the instalment structure on Ray Co., given that it would pay optimally under the new structure? The effect of instalment liability being based solely on preceding years (on b_1 and b_2), is that Ray Co.'s AETR was 2.93% over the benchmark rate. It is important however to note that AETR's are very sensitive to b_1; for example, if b_1 were decreased from $12,000,000 (167% of the expected value of tax liability for the year) to $4,800,000 (67% of the expected value of tax liability for the year), the AETR would decline to 43.35 (2.14% less than the benchmark rate). That is, this change in b_1 would lead to a percentage reduction in AETR of more than 5% (2.93-(-2.14)).

The government, through the timing of refunds, can affect the size of the stub loss and therefore affect a corporation's AETR and METR. With electronic filing and direct depositing of refunds, it is becoming technically possible for Revenue Canada to provide tax refunds almost immediately. If the government were to provide immediate refunds, on the remainder due date, the effect on Ray Co. would be small (the change in AETR and METR would be less than 1/10th of 1%). The effect of the government paying on the date just such that it would not incur interest payments to the corporation, would similarly be small.

Eliminating the penalty provision would similarly have little effect on Ray Co.; the change in AETR and METR would be less than 1/10th of 1%.

What would be the impact of eliminating the instalment structure; that is, requiring payment only at the remainder due date? An equivalent question is, what would be the equivalent AETR and METR to the statutory rate if a corporation was not required to pay instalments? As one may expect, the impact would be significant. The percentage decline in average and marginal effective tax rates is 4.28%; a decrease from the statutory rate of 44.3% to 42.4%.

Recall that the above rate effects on Ray Co. of changing the instalment structure were based on it paying instalments optimally. In the following sections, it is demonstrated that the magnitude of AETR's and METR's may change significantly where corporations do not pay optimally, and across corporations.

6.3.2 Sub-optimal Payment Strategies and Horizontal Equity

Horizontal equity, that "equals should be treated equally", is an important criterion in formulating tax policy.[63] If corporations are unequally informed with respect to the optimal payment of instalments, there may be a potential for horizontal inequity; that is, there may be a differential tax on the uninformed. Payment strategies presented by tax practitioners (as set out in chapter 1), are sub-optimal relative to those set out in this thesis. The cost of being uninformed as to paying optimally is therefore the additional expected loss from following these "naive" strategies. The larger the difference between the expected loss across payment strategies, the greater the potential horizontal inequities.

[63]For a more complete discussion of horizontal equity, read Feldstein [1976], Musgrave [1976], or Kaplow [1989].

As noted in the introduction, several strategies for payment have been discussed in the professional literature. These include: paying the expected value of the corporation's tax liability for the year in 12 equal payments; paying the corporation's first instalment base in 12 equal payments; paying based on the corporation's second instalment base for the first two payments, then paying based on the first instalment base for the remaining 10 payments; and paying zero in the year.

To illustrate the magnitude of horizontal inequities from corporation's following different payment strategies, let us revisit the Ray Co. example above. In Table 6.2, Ray Co.'s expected loss, AETR, and percentage change in AETR from the benchmark rate will be presented. In evaluating the size of the expected loss, recall that Ray Co's expected tax liability for the year is $7.2 million. Note that AETR is generally considered to be the appropriate measure for examining horizontal equity.[64] To help compare the non-optimal rates to the optimal rates, the percentage increase in the present value of all payments to Revenue Canada from following a non-optimal payment strategy over the optimal payment strategy,

$$\alpha = \frac{E(t_{pv})^{non\text{-}opt} - E(t_{pv})^{opt}}{E(t_{pv})^{opt}} \tag{6.10}$$

is also presented.

[64]Callihan [1992, 1] notes that the appropriate measure to assess equity or neutrality of the tax system are AETR's.

TABLE 6.2

Ray Co.'s Expected Loss, α, AETR and Percentage Change in AETR from the Benchmark Structure Following Various Payment Strategies

Payment Strategy	Expected Loss	α	AETR	%Δ AETR
Optimal Payments (Chapter 5)	$65,935	-	44.64	.77
Pay 1/12 of expected value of tax liability, $E(x)$, each month	$122,250	.74	44.97	1.52
Pay 1/12 of first instalment base, b_1, each month	$262,095	2.59	45.80	3.38
Pay based on second instalment base, b_2, for first two months, then pay based on first instalment base, b_1	$247,383	2.39	45.71	3.18
Pay zero	$440,264	4.94	46.84	5.74

Table 5.3 illustrates the magnitude of potential costs to corporations of not following optimal strategies. Note that following any of the non-optimal strategies listed in Table 5.3 would significantly increase Ray Co.'s expected loss, α, and AETR. For example, if Ray Co. paid based on its first instalment base, its expected loss would be $196,160 greater than if it had followed the optimal strategy; this would have the effect of increasing its expected tax liability for the year by 2.588%, with the corporation's AETR 3.376% greater than the benchmark rate.[65]

Note that the AETR's are significantly larger where Ray Co. does not pay optimally than where it does. If a substantial number of corporations are not paying instalments optimally, the effect on effective tax rates relative to the benchmark may be much larger than one may otherwise expect. That is, effective tax rate measures are very sensitive to rationality assumptions.

6.3.3 The Effect, on a Corporation, of Differing Parameter Values

Even where corporations pay optimally, horizontal inequities may be created through the instalment structure. That is, corporations with different costs of capital, different tax liability histories, and different levels of uncertainty, will have different expected losses (effective tax rate effects). These are illustrated in Table 6.3.

[65]Note that the ranking of expected losses across naive strategies depends on the specific parameter values (*i.e.*, for certain parameter values, the "naive" strategy which provides the smallest expected loss would be paying zero).

TABLE 6.3

Ray Co's AETR and METR and Percentage Changes in these Rates from the Statutory Rate (44.3%) For Different Parameter Values

Parameter Values[66]	AETR	%Δ AETR	METR	%Δ METR
1. The Corporation's Cost of Capital				
a. 5%	44.59	.65	44.25	-.10
b. 7%	44.64	.77	44.22	-.17
c. 9%	44.60	.67	44.17	-.30
2. Preceding Year's Tax Liability, b_1				
a. $14,400,000	44.70	.90	44.27	-.07
b. $10,800,000	44.60	.67	43.09	-2.73
c. $7,200,000	44.17	-.30	42.45	-4.17
d. $3,600,000	43.35	-2.15	42.40	-4.28
e. $0	42.40	-4.28	42.40	-4.28
3. Instalment Liability				
a. $Prob_{uuuu} = Prob_{dddd} = .25$; all other states equally likely	44.39	.19	43.42	-.20
b. All states equally likely	44.64	.77	44.22	-.17
c. $Prob_{uuuu} = Prob_{dddd} = .001$; all other states equally likely	44.65	.80	44.55	-.55

Note that the corporation's cost of capital had little effect on the corporation's AETR and METR. The preceding year's tax liability, b_1, has a significant effect on both Ray Co.'s AETR and METR; the larger is b_1, the larger is the corporations AETR and METR.

[66]Note that 1b. and 3b. are the base case from row 1 in Table 6.1.

Changing the probabilities of certain states, such that the expected loss remains constant but variance increases or decreases, a moderate change in rates is observed.

The level of horizontal inequity associated with corporation's being uninformed, if it is considered to be a significant problem, may be reduced in the following ways. First, Revenue Canada could attempt to increase the level of knowledge with respect to optimal tax planning in the present system. Providing assistance of this form would be a departure from the present to the extent that tax planning advice is offered by Revenue Canada. Second, the instalment structure could be modified to reduce horizontal inequities resulting from unequal information. One such reform would be to base instalments solely on preceding year's tax liability. This would eliminate uncertainty in the problem.

CHAPTER 7

CONCLUSION

7.1 Concluding Remarks

General contributions of this thesis are discussed below, followed by discussions of specific contributions.

Countries utilizing instalment structures to collect corporate tax are on every continent, from every political system, and at every stage of economic development. In the Canadian context almost all corporations with positive tax liabilities must pay tax instalments. The study of the Canadian corporate instalment structure is therefore important in its broad applicability.

The Canadian corporate instalment structure is complex (as are structures in other countries such as the United States). Resulting from this complexity, the practitioner literature is full of heuristics and general statements concerning the payment of instalments. An incentive theory as to how corporation's should pay instalments given these complex instalment structures does not exist. This thesis, in developing models to determine the optimal timing of instalment payments, provides such a theory.

Specific contributions of this thesis are discussed below in the order they were developed. In chapter 2, the corporation's loss from instalments was developed. In formulating this loss, provisions of the Income Tax Act, and the corporation's opportunity losses and gains, were modelled. Note that the professional literature has largely ignored the corporation's cost of capital. That is, that literature focuses on payments made to the government for underpaying instalments (including the penalty for substantial underpayment), and not on losses based on the time value of money (such as the stub loss). The development of analytic expressions for the corporation's instalment liability for tax years prior to 1992 and for years after 1991 further contributes to the tax literature: these expressions provide useful insights into the instalment structure. For example, for tax years prior to 1992 switching between instalment alternatives in the year was possible, although this was not previously recognized as a possibility in the professional or government literature. The conditions under which switching would occur, and its implications, were developed in Appendix A. Finally, an alternative "equivalent" objective function in which the form of the present value of cash flows was developed.

Chapters 3 and 4 developed analytic models for a risk neutral corporation which minimizes its expected instalment loss. To illustrate certain non-time related tradeoffs, single period models were developed in chapter 3. The tradeoff between underpayment and overpayment was first established without the penalty structure. The effects of the penalty structure, with the associated "kinks" in the expected loss function, were then examined.

Chapter 4 examined analytic consequences of the multi-period (12 monthly payment) structure. In this model, uncertainty was presented as evolving over time (*i.e.*, a corporation would have better information about its tax liability for the year in the fourth quarter than in the first quarter). Using Diewert conditions (one-sided directional derivatives), and restricting the problem such that certain effects could be isolated, the following analytic results were developed. First, the effect of the evolution of uncertainty was examined where interest rates were simple and constant, and without stub loss. The corporation, under these conditions, would make a single large payment (generally several times its expected tax liability for the year), in December: that is, as information reduces the probability of overpaying, and as there is no cost to delaying payment under the stated assumptions, the corporation would not make a payment prior to the last payment date. Second, the effect of introducing the stub loss was examined under the assumptions that the corporation's tax liability for the year was certain, and rates were simple and unchanging. The stub loss created incentive for the corporation not to delay payments as much; *i.e.*, for certain payment paths, a cost would be imposed on paying instalments late (the corporation could incur interest from underpayment and a stub loss). Third, extending this second result through relaxing the simple interest assumption (that is, allowing compound interest), the corporation would make a single payment (generally much smaller than its expected tax liability for the year) at the first payment date. Note however, that this compounding affect was relatively small.

Numerical optimization in the form of linear programming was utilized in chapter 5 to determine the corporation's optimal contingent payment vector. The formulation did not impose the restrictions necessary for analytic solutions in chapter 4. As the expected loss function developed in chapter 4 had not allowed for non-deterministic rates (although it had allowed rates to change), expected loss and linear programming formulations were developed to allow this inclusion. A discussion of the application of the linear programming models, and an example of the implementation of numerical optimization was then provided.

Chapter 6 focused on the development of methods to examine certain policy implications of the instalment structure. A measure of the percentage difference in tax from a benchmark structure, a measure of the effect of instalments on effective tax rates, was developed. Potential applications of this method were illustrated using an example. The effect of sub-optimal payment strategies on horizontal equity was then examined.

7.2 Directions for Future Research

Future research includes application of the results and methods developed in this thesis in several directions.

(1). A first direction for future research is the application of the methods developed in this thesis to personal income taxes and to other countries; for example, to the United States corporate instalment structure. The United States structure, although similar to the Canadian structure in many respects, has significant differences. One important difference is that the United States structure allows only partial offsetting of interest (*i.e.*, interest owing from underpayments may not be offset by overpayments at a later date). Another important difference is the lack of an equivalent penalty structure in the United States. Other differences, such as quarterly rather than monthly payments, may be important in determining the impact of the instalment structure on effective tax rates.

(2). A corporation's loss from instalment payments is a function of its tax liability for the year which, in this thesis, is assumed to be exogenous. Strategies which affect the corporation's tax liability for the year such as shifting income between years, and filing strategically, have therefore not been examined. It would however, appear to be a worthwhile exercise to have the corporation jointly determine its instalment payment strategy and its filing or income-shifting strategy.

Corporation's may shift income between years (*i.e.*, income smoothing) through, for example, not claiming capital cost allowance or advancing or delaying capital purchases. The marginal cost of shifting income between years will differ across instalment outcomes: for example, recall Figure 4.1. Decreasing the tax liability for the year by income shifting produces a leftward shift in the loss, *i.e.*, from $l(\omega_2)$ to $l(\omega_1)$. Hence, if the corporation is in a penalty position without stub loss, the cost to the corporation of deferring income such that its tax liability is reduced by $1 is $1.5g_i$ less than if the corporation had neither overpaid nor underpaid instalments. Therefore, the corporation has a greater incentive to decrease income if it discovers at the remainder due date that it is in a penalty position.

(3). At the end of section 2.2.4 in chapter 2, there is some discussion of the possibility of a corporation shifting instalment payments to another account if it discovers that it has overpaid. A more formal modelling of this possibility would be desirable. For example, if the cost of underpaying employee source deduction accounts was less than the cost of underpaying instalment accounts, it might affect the optimal instalment payment strategy.

(4). The central focus of the tax compliance literature is the taxpayer's optimal filing decision given the taxpayer and the revenue authority have asymmetric information concerning the true tax liability for the year. That is, it will be optimal given certain parameter values for a taxpayer to report an amount less than X (even if it is known by him or her with certainty). An important extension would therefore be an examination of the interrelationship between a corporation's instalment payment strategy and its reporting strategy. Note that empirical literature demonstrates those who have underpaid instalments are more likely to report falsely. Christian, Gupta, and Willis [1993] examine empirically the effect of a taxpayer being in a refund or balance due position on compliance (as had Chang and Schultz [1990] previously). It would appear that an extension of the models in this thesis may provide an analytic justification for these results.

(5). It is assumed throughout this thesis that corporations are risk neutral; that they pay instalments such that they minimize the expected value of their loss. Although it appears that the introduction of risk aversion into the models developed in this thesis would add significant complexity, it may be worthwhile to examine potential effects in simplified settings. Note that as the models developed trade-off costs associated with overpayment and underpayment, whether risk aversion increases or decreases a particular payment will depend on the specific parameters.

(6). This thesis does not explicitly model the relationship between the corporation's cost of capital and its tax liability for the year: that is, changing the amount (or timing) of instalments will affect the amount of interest deduction in the year and consequently the corporation's tax liability for the year, X. The effect is that X is a function of p. This relationship could be formalized. Although it is unlikely that this would have a significant effect on instalment payments, it may be important for capital budgeting.[67]

[67] Mumey and Sick [1990] studies the effect of a simplified instalment structure on capital budgeting.

(7). The effect of interest rates which are either stochastic or deterministic but changing over time on a corporation's optimal instalment payment strategy has not been examined analytically. There are two primary complications which arise from non-constant rates which it would be interesting to analyze. First, the expectation of future rates may affect the optimal solution even in the absence of a lag in the rate g_i. That is, to the extent that the corporation has formed expectations that the relative cost of underpayment and overpayment will change over time, these expectations will affect the optimal time path of payments. Second, as g_i is a lagged variable, the rate is determined using 3-month T-Bill rates from the prior quarter, additional effects related to this lag may be examined.

(8). Finally, although it would be beneficial to examine the behaviour of taxpayers with respect to optimal payment strategy, it is difficult to obtain access to detailed tax information[68]. Note that this does not preclude Revenue Canada or the IRS from undertaking such studies. Further, revenue authorities are now permitting some limited use of such data,[69] and it may be possible to determine the instalment payments made in a quarter from financial accounting statements.[70]

[68]See Macnaughton [1992].

[69]For example, see Shackelford, Collins, and Kemsley [1994].

[70]See Reimer [1994].

APPENDIX A

THE CALCULATION OF INSTALMENT LIABILITY FOR FISCAL YEARS PRIOR TO 1992

In June of 1994 an amendment to subsection 161(4.1) which applies to 1992 and subsequent years became law. Recall that subsection 157(1) provides three alternative methods for calculating instalment payments. Subsection 161(4.1), prior to this amendment, placed further structure on these calculations such that a corporation is liable to pay the least amount required to be paid (under the three alternatives set out in subsection 157(1)) *on or before each* of the payment dates. Recall that the new wording is that the taxpayer is liable to pay *the least total amount of instalments of tax for the year*. That is, whereas the aggregate instalment liability at any payment date prior to this change was the least of the total amounts owing to that date under each of the three alternatives, it is now based on the single alternative which gives the lowest total instalments for the year.

The aggregate instalment liability, under the previous law, at any payment date i can be expressed as,

$$\sum_{j=1}^{i} q_j = \min(I_i, II_i, III_i) \qquad \text{(A.1)}$$

where q_j is the instalment liability for month j and the three alternative amounts are as follows:

I_i. an amount equal to i instalments of 1/12 of the estimated tax payable;

II_i. an amount equal to i instalments of 1/12 of the corporation's first instalment base; or

III_i. for the first two instalments, an amount equal to i multiplied by 1/12 of the second instalment base, and for subsequent instalments, i-2 instalments of 1/10 of the remainder of the first instalment base (that is, after deducting the first 2 instalments from the first instalment base).

Algebraically, these amounts may be written,[71]

$$\begin{aligned} I_i &= i\,x && \text{for } i = 1,2,\ldots,12 \\ II_i &= i\,b_1 && \text{for } i = 1,2,\ldots,12 \\ III_i &= \begin{cases} i b_2 & \text{for } i = 1,2 \\ 2b_2 + (i-2)\frac{1}{10}(12b_1 - 2b_2) & \text{for } i = 3,4,\ldots,12 \end{cases} \end{aligned} \qquad \text{(A.2)}$$

where (as in the text) x is 1/12 of the corporation's tax liability for the year, and b_1 and b_2 are

[71]For alternative III, this formulation assumes that $2b_2 \leq 12b_1$, *i.e.*, the sum of the first two instalments under this method does not exceed the first instalment base. This assumption has no effect on the results.

1/12 of the corporation's first and second instalment bases for the year respectively.

Using equation (A.1) above, let us define the instalment liability for month j as the increment in the aggregate instalment liability from payment date j-1 to payment date j:

$$q_j = \begin{cases} \min(I_j, II_j, III_j) & \text{for } j = 1 \\ \min(I_j, II_j, III_j) - \min(I_{j-1}, II_{j-1}, III_{j-1}) & \text{for } j = 2 \text{ to } 12 \end{cases} \tag{A.3}$$

Substituting equation (A.2) into equation (A.3), the following expression for instalment liability in any month j is derived,

$$q_j = \begin{cases} \min(x, b_1, b_2) & \text{for } j = 1,2 \\ \min\left(jx, jb_1, 2b_2 + (j-2)\frac{1}{10}(12b_1 - 2b_2)\right) & \\ \quad - \min\left((j-1)x, (j-1)b_1, 2b_2 + (j-3)\frac{1}{10}(12b_1 - 2b_2)\right) & \text{for } j = 3 \text{ to } 12 \end{cases} \tag{A.4}$$

Note that q_j is exogenous as the corporation's tax liability for the year and its first and second instalment bases are exogenous in this model.

Switching between the Three Alternatives set out in Subsection 157(1)

Neither Information Circular 81-11R3 nor the academic or professional literature explicitly recognized that the minimum of the three payment alternatives would change, under certain circumstances, across payment dates. That is, the least of the three alternatives, *I*, *II*, and *III*, at some payment date i may not have been the least of the three alternatives at another payment date j. It is worthwhile to define the conditions under which the minimum alternative changed, and where it remained constant over time. This information would have been important to corporations as, without understanding the instalment liability structure, significant errors in payment strategy may have arisen.

The following section demonstrates that the aggregate instalment liability for tax years prior to 1992 may be rewritten as,

$$\sum_{j=1}^{i} q_j = \begin{cases} I_i & \forall\ i = 1 \text{ to } 12 & \text{if } \ x \leq \{b_1, b_2\} \\ II_i & \forall\ i = 1 \text{ to } 12 & \text{if } \ b_1 \leq \{x, b_2\} \\ III_i & \forall\ i = 1 \text{ to } 12 & \text{if } \ b_2 \leq b_1 \leq x \\ \begin{Bmatrix} III_i & \forall\ i < k \\ I_i & \forall\ i \geq k \end{Bmatrix} & & \text{if } \ b_2 \leq x \leq b_1 \end{cases} \tag{A.5}$$

where k is the smallest integer such that,

$$k \geq \frac{12(b_2 - b_1)}{5x + b_2 - 6b_1} \tag{A.6}$$

Note that the only case in which there will be switching is where $b_2 \leq x \leq b_1$. That is, where a corporation's second instalment base is less than its tax liability for the year which in turn is less than its first instalment base, its instalment liability will initially be based on alternative *III* (for at least the first two periods), but will always switch to alternative *I* at or before the twelfth payment date.

The following expression for q_i, which is also derived, provides an expression for q_i

over the different possible orderings of x, b_1, and b_2,

$$q_i = \begin{cases} x & \forall\ i = 1 \text{ to } 12 & \text{if } x \le \{b_1, b_2\} \\ b_1 & \forall\ i = 1 \text{ to } 12 & \text{if } b_1 \le \{x, b_2\} \\ \left\{ \begin{array}{ll} b_2 & \forall\ i = 1,2 \\ \frac{1}{10}(12b_1 - 2b_2) & \forall\ i = 3 \text{ to } 12 \end{array} \right\} & & \text{if } b_2 \le b_1 \le x \\ \left\{ \begin{array}{ll} b_2 & \forall\ i = 1,2 \\ \frac{1}{10}(12b_1 - 2b_2) & \forall\ i < k \\ ix - 2b_2 + (i-2)\frac{1}{10}(12b_1 - 2b_2) & \text{for 1st } i \ge k \\ x & \forall\ i \ge k+1 \end{array} \right\} & & \text{if } b_2 \le x \le b_1 \end{cases} \quad \text{(A.7)}$$

where payment date k is as defined in equation (A.6). This expression, in setting out the liability path for each possible ordering of x, b_1, and b_2, is useful to corporations and to Revenue Canada. In any fiscal year, the corporation's liability would follow one of the four paths defined in equation (A.7).

The Derivation of q_i for Years Prior to 1992

As x, b_1, and b_2 are constant across months in a given fiscal year, only six relationships can occur:

1. $x \leq b_1 \leq b_2$
2. $x \leq b_2 \leq b_1$
3. $b_1 \leq b_2 \leq x$
4. $b_1 \leq x \leq b_2$
5. $b_2 \leq b_1 \leq x$
6. $b_2 \leq x \leq b_1$

Instalment liability for these six relationships is examined in the following four propositions.

Proposition A1:

Where x is less than or equal to both b_1 and b_2 (relationships 1. and 2. above), $\min(I_i, II_i, III_i) = I_i$ for $i = 1$ to 12.

Proof:

Equation (A.1) defines the aggregate instalment liability at any payment date i as,

$$\sum_{j=1}^{i} q_j = \min\,(I_i, II_i, III_i)\,. \tag{A.8}$$

Substituting the values for I_i, II_i, and III_i from equation (A.2) into equation (A.8), provides the following expression for the aggregate instalment liability at any payment date i,

$$\sum_{j=1}^{i} q_j = \begin{cases} \min(i\,x,\, i\,b_1,\, i\,b_2) & \textit{for} \quad i = 1,2 \\ \min\left(i\,x,\, i\,b_1,\, 2b_2 + (i-2)\frac{1}{10}(12b_1 - 2b_2)\right) & \textit{for} \quad i = 3 \textit{ to } 12 \end{cases} \tag{A.9}$$

If x is less than or equal to b_1, ix must be less than or equal to ib_1, for all i (as i is strictly positive); that is, $\min(I_i, II_i) = I_i$ for $i = 1$ to 12. Similarly, if x is less than or equal to b_2, then ix is less than or equal to ib_2 for $i=1,2$ (*i.e.*, $\min(I_i, III_i) = I_i$ for $i = 1,2$). It therefore only remains to be shown that;

$$
\begin{aligned}
&\min(I_i, II_i) = I_i \qquad\qquad for \quad i = 3 \ to \ 12 \\
&\rightarrow \ ix \leq 2b_2 + (i-2)\frac{1}{10}(12b_1 - 2b_2) \\
&\rightarrow \ ix \leq \left(\frac{12}{10}(i-2)\right)b_1 + \left(\frac{2}{10}(12-i)\right)b_2 \\
&\rightarrow \ x \leq \frac{\left(\frac{12}{10}(i-2)\right)}{i}b_1 + \frac{\left(\frac{2}{10}(12-i)\right)}{i}b_2 \\
&\rightarrow \ x \leq \alpha b_1 + (1-\alpha)b_2 \quad where \quad \alpha = \frac{\frac{12}{10}(i-2)}{i}
\end{aligned}
\qquad \text{(A.10)}
$$

As $0 \leq \alpha \leq 1$, and the weights on b_1 and b_2 sum to 1, then given $x \leq \{b_1, b_2\}$, a linear combination of b_1 and b_2 is at least as large as x and therefore $\min(I_i, III_i) = I_i$ for $i = 3$ to 12.

Proposition A2:

Where b_1 is less than or equal to both x and b_2 (relationships 3. and 4. above), $\min(I_i, II_i, III_i) = II_i$ for $i = 1$ to 12. The proof of this proposition is substantially the same as the proof of Proposition A1 above.

Proposition A3:

Where b_2 is less than or equal to x which is less than or equal to b_1 (relationship 6. above), there exists one integer j such that $3 \leq j \leq 12$ and,

$$\min(I_i, II_i, III_i) = \begin{cases} III_i & \forall\ i < j \\ I_i & \forall\ i \geq j \end{cases} \tag{A.11}$$

where j is the smallest integer such that,

$$j \geq \frac{12(b_2 - b_1)}{5x + b_2 - 6b_1} \tag{A.12}$$

Restating, where $b_2 \leq x \leq b_1$, $\min(I_i, II_i, III_i) = III_i$ for $i=1,\ldots j\text{-}1$ (the corporation's instalment liability for the first j-1 periods is determined by III_i), and $\min(I_i, II_i, III_i) = I_i$ for $i = j, \ldots, 12$ (the corporation's instalment liability for the periods j to the final period 12 is determined by I_i).

Proof:

If x is less than or equal to b_1, ix must be less than or equal to ib_1, for all i (as i is strictly positive); that is, $\min(I_i, II_i) = I_i$ for $i = 1$ to 12. Therefore, alternative II need not be considered further.

Also, as b_2 is less than or equal to x, then ib_1 is less than or equal to ix for $i=1,2$ (*i.e.*, $\min(I_i, III_i) = III_i$ for $i = 1,2$). Therefore, $\min(I_i, II_i, III_i) = III_i$ for $i = 1,2$.

For $i = 12$,

$$\begin{aligned} \min(I_i, II_i, III_i) &= \min(I_i, III_i) \\ &= \min(12x, 12b_1) \\ &= 12b_1 \qquad \textit{since} \quad b_1 \le x \\ &= I_i \end{aligned} \tag{A.13}$$

Therefore, at payment dates 1 and 2 the minimum instalment liability will always be calculated using III_i, and at time 12, the minimum instalment liability will always be calculated using I_i.

To complete the proof of Proposition A3, we need only prove that the instalment liability can only switch from III_i to I_i once, and to determine the date at which the liability will shift. From equation (A.9), the derivative of the difference between III_i and I_i with respect to i for $i > 2$ is,

$$\begin{aligned} \frac{d(III_i - I_i)}{di} &= (1.2b_1 - 0.2b_2) - x \\ &\ge 1.2b_1 - 0.2b_1 - x \qquad \textit{as} \quad b_2 \le b_1 \\ &\ge b_1 - x \\ &\ge 0 \qquad \textit{as} \quad x \le b_1 \end{aligned}$$

which indicates that instalment liability can only switch from III_i and I_i once in the year.

From (A.9), the crossing point for a continuous j would be,

$$
\begin{aligned}
I_j &= III_j \\
\Rightarrow\ jx &= 2b_2 + (i-2)\frac{1}{10}(12b_1 - 2b_2) \\
\Rightarrow\ j(x - \frac{1}{10}(12b_1 - 2b_2)) &= 2b_2 - \frac{2}{10}(12b_1 - 2b_2) \qquad \text{(A.15)} \\
\Rightarrow\ j &= \frac{12(b_2 - b_1)}{5x + b_2 - 6b_1}
\end{aligned}
$$

Since j must be an integer, choose the smallest integer which is greater than or equal to the amount determined in equation (A.15).

Proposition A4:

Where b_2 is less than or equal to x which is less than or equal to b_2 (relationships 5. above), $\min(I_i, II_i, III_i) = III_i$ for i = 1 to 12.

Proof:

If b_1 is less than or equal to x, ib_1 must be less than or equal to ix for all i; that is, $\min(I_i, II_i) = II_i$ for i = 1 to 12. Therefore, alternative I need not be considered further. Also, as b_2 is less than or equal to b_1, then ib_2 is less than or equal to ib_1 for i=1,2 (*i.e.*, $\min(II_i, III_i) = III_i$ for i = 1,2).

Alternative II is steadily improving relative to alternative III throughout the period $i = 3,4,\ldots,12$:

$$\frac{d(III_i - II_i)}{di} = (1.2b_1 - 0.2b_2) - b_1 \tag{A.16}$$

$$= (0.2b_1 - 0.2b_2) \geq 0 \qquad as \;\; b_1 \geq b_2$$

However, alternative III never overtakes alternative II as instalment liability is equal under these two alternatives at date 12:

$$\min(II_i, III_i) = (12b_1, 12b_1) \tag{A.17}$$

Therefore, $\min(I_i, II_i, III_i) = III_i$ for $i = 1$ to 12.

Q.E.D.

It is useful to summarize the results in three equivalent ways: first, stating the aggregate instalment liability, $\sum_{k=1}^{i} q_k$, in terms of I_i, II_i, and III_i; second, stating the aggregate instalment liability in terms of x, b_1, and b_2; and third, stating the instalment liability for each period i, q_i, in terms of x, b_1, and b_2.

Substituting the results of propositions A1 through A4 into equation (A.9) gives,

$$\sum_{k=1}^{i} q_k = \begin{cases} I_i & \forall\ i = 1 \text{ to } 12 & \text{if } x \le \{b_1, b_2\} \\ II_i & \forall\ i = 1 \text{ to } 12 & \text{if } b_1 \le \{x, b_2\} \\ III_i & \forall\ i = 1 \text{ to } 12 & \text{if } b_2 \le b_1 \le x \\ III_i & \forall\ i < j & \text{if } b_2 \le x \le b_1 \\ I_i & \forall\ i \ge j & \end{cases} \tag{A.18}$$

where j is the smallest integer such that,

$$j \ge \frac{12(b_2 - b_1)}{5x + b_2 - 6b_1} \tag{A.19}$$

An equivalent expression in terms of x, b_1, and b_2 may be written,

$$\sum_{k=1}^{i} q_k = \begin{cases} ix & \forall\ i = 1 \text{ to } 12 & \text{if } x \le \{b_1, b_2\} \\ ib_1 & \forall\ i = 1 \text{ to } 12 & \text{if } b_1 \le \{x, b_2\} \\ ib_2 & \forall\ i = 1,2 & \\ 2b_2 + (i-2)\frac{1}{10}(12b_1 - 2b_2) & \forall\ i = 3 \text{ to } 12 & \text{if } b_2 \le b_1 \le x \\ ib_2 & \forall\ i = 1,2 & \\ 2b_2 + (i-2)\frac{1}{10}(12b_1 - 2b_2) & \forall\ i < j & \text{if } b_2 \le x \le b_1 \\ ix & \forall\ i \ge j & \end{cases} \tag{A.20}$$

The corporation's instalment liability for month i, from equation (2.4), is,

$$q_i = \begin{cases} \min(I_i, II_i, III_i) & \text{for } i = 1 \\ \min(I_i, II_i, III_i) - \min(I_{i-1}, II_{i-1}, III_{i-1}) & \text{for } i = 2 \text{ to } 12 \end{cases} \tag{A.21}$$

Substituting equation (A.20) into equation (A.21), the following expression may be derived for q_i;

$$q_i = \begin{cases} x & \forall\ i = 1 \text{ to } 12 \quad \text{if } x \le \{b_1, b_2\} \\ b_1 & \forall\ i = 1 \text{ to } 12 \quad \text{if } b_1 \le \{x, b_2\} \\ \left\{ \begin{array}{ll} b_2 & \forall\ i = 1,2 \\ \frac{1}{10}(12\,b_1 - 2\,b_2) & \forall\ i = 3 \text{ to } 12 \end{array} \right\} & \text{if } b_2 \le b_1 \le x \\ \left\{ \begin{array}{ll} b_2 & \forall\ i = 1,2 \\ \frac{1}{10}(12\,b_1 - 2\,b_2) & \forall\ i < j \\ i\,x - 2b_2 + (i-2)\frac{1}{10}(12b_1 - 2b_2) & \text{for 1st } i \ge j \\ x & \forall\ i \ge j+1 \end{array} \right\} & \text{if } b_2 \le x \le b_1 \end{cases} \tag{A.22}$$

where j is defined in equation (A.12).

APPENDIX B

DETERMINING THE CORPORATION'S INSTALMENT INTEREST

This appendix serves two purposes. First, it demonstrates that the amount of instalment interest payable by the corporation, U, is always the amount determined under subsection 161(2.2); that is, the amount payable under subsection 161(2.2) is always less than or equal to that determined under subsection 161(2). Second, equivalent formulations for U are derived. The first equivalent formulation is developed in proving that the amount determined under subsection 161(2.2) dominates (is always less than or equal to) the amount determined under subsection 161(2). It is then demonstrated that a second method, the calculation method used in Information Circular 81-11R, is in fact equivalent. Finally, using an example from Information Circular 81-11R, computation of U using each of the three alternative methods is demonstrated.

Proposition B1:

The amount of instalment interest payable by the corporation under 161(2.2) is always less than or equal to the amount determined under 161(2); that is 161(2.2) $\leq$ 161(2).

Proof:

The proof of Proposition B1 takes three steps: first, an alternative expression for 161(2.2) is developed; second, an expression for 161(2) is developed; and third, it is demonstrated that the alternative expression for 161(2.2) is less than or equal to the expression for 161(2).

Step 1:

Recall that the amount determined under 161(2.2), equation (2.12), is

$$U = \max\left[0,\ \sum_{i=1}^{12} (q_i - p_i)\, g_i\right] \tag{B.1}$$

where

$$g_i = \prod_{k=i+1}^{13} \left(1 + \frac{G_k}{365}\right)^{N_k} - 1$$

It is demonstrated that the following expression is equivalent to equation (B.1),

$$U = \max\left[0,\ \sum_{i=1}^{12} \left(\sum_{j=1}^{i} q_j - \sum_{j=1}^{i} p_j\right) \bar{g}_i\right] \tag{B.2}$$

where

$$\bar{g}_i = \left[\left(1 + \frac{G_{i+1}}{365}\right)^{N_{i+1}} - 1\right] \cdot \prod_{k=i+2}^{13} \left(1 + \frac{G_k}{365}\right)^{N_k}$$

Before presenting the proof, let us interpret the meaning of equation (B.2). Note that this expression is based on the difference between the cumulative amount owing, $\sum_{j=1}^{i} q_j$, and the cumulative amount paid, $\sum_{j=1}^{i} p_i$, up to and including payment date i. Interest is calculated on the cumulative overpayment or underpayment amount at each payment date for the period (a month or the stub period) $i+1$, which is then taken forward to the remainder due date. The expression in square brackets in $\overline{g_i}$ represents the amount of interest on one dollar of underpayment in the instalment period (or contra interest on one dollar of overpayment) for the period $i+1$. The remainder of the expression (the right hand side) brings this amount forward to the remainder due date. Therefore, the amount determined under equation (B.2) is simply the sum of the amounts of interest on cumulative overpayments and underpayments at each payment date i.

To begin the proof, not that equations (B.1) and (B.2) must be equivalent,

$$\max\left[0, \sum_{i=1}^{12} (q_i - p_i) g_i\right] = \max\left[0, \sum_{i=1}^{12} \left(\sum_{j=1}^{i} q_j - \sum_{j=1}^{i} p_j\right) \overline{g_i}\right] \quad \textbf{(B.4)}$$

if it can be shown that,

$$\sum_{i=1}^{12} (q_i - p_i) g_i = \sum_{i=1}^{12} \left(\sum_{j=1}^{i} q_j - \sum_{j=1}^{i} p_j\right) \overline{g_i} \quad \textbf{(B.5)}$$

The right hand side of equation (B.5) may be rewritten, through rearranging terms, as,

$$\sum_{i=1}^{12} \sum_{j=1}^{i} (q_j - p_j)\overline{g_i} \tag{B.6}$$

Expanding this expression into 12 terms, each corresponding to a particular value of i, gives

$$\begin{aligned}\sum_{i=1}^{12} \sum_{j=1}^{i} (q_j - p_j)\overline{g_i} = & (q_1 - p_1)\left[\overline{g_1} + \overline{g_2} + \overline{g_3} + \dots + \overline{g_{12}}\right] \\ & + (q_2 - p_2)\left[\overline{g_2} + \overline{g_3} + \dots + \overline{g_{12}}\right] \\ & + (q_3 - p_3)\left[\overline{g_3} + \overline{g_4} + \dots + \overline{g_{12}}\right] \\ & + \quad \dots \\ & + (q_{12} - p_{12})\overline{g_{12}}\end{aligned} \tag{B.7}$$

Further, since

$$\sum_{j=1}^{12} \overline{g_i} = \prod_{k=i+1}^{13} \left(1 + \frac{G_k}{365}\right)^{N_k} - 1 = g_i \tag{B.8}$$

equation (B.7) may be rewritten,

$$\sum_{i=1}^{12} (q_i - p_i)\, g_i \tag{B.9}$$

which is the left hand side of equation (B.5). Therefore, it follows that equations (B.1) and (B.2) are equal.

<u>Step 2:</u>

Subsection 161(2) states that where a corporation has failed to pay "on or before the day on or before which the ... instalment ... was required to be paid, he shall pay ... interest... on the amount computed from the day on or before which the amount was required to be paid". Subsection 161(4.1) requires that for the purposes of subsection 161(2) the corporation is "deemed to have been liable to pay" by reference to subsection 157(1) by "whichever method gives rise to the least amount required to be paid by the corporation on or before" the dates set out in that subsection. The amount which is required to be paid on the last day of each month is therefore the cumulative liability to that date i, $\sum_{j=1}^{i} q_j$, less the aggregate of all payments to that date, $\sum_{j=1}^{i} p_j$. The interest on that amount from that payment date to the following payment date (taken forward to the remainder due date) may therefore be written,

$$\left(\sum_{j=1}^{i} q_j - \sum_{j=1}^{i} p_j\right)\overline{g_i} \tag{B.10}$$

For each payment date i, as this amount is the interest on the amount which the corporation failed to pay on or before that date, it may be written,

$$\max\left[0, \left(\sum_{j=1}^{i} q_j - \sum_{j=1}^{i} p_j\right)\overline{g_i}\right] \tag{B.11}$$

Summing over the 12 monthly payments, the amount determined under subsection 161(2) equals,

$$161(2) = \sum_{i=1}^{12} \max\left[0, \left(\sum_{j=1}^{i} q_j - \sum_{j=1}^{i} p_j\right)\overline{g_i}\right] \tag{B.12}$$

<u>Step 3:</u>

To demonstrate that subsection 161(2.2) always dominates subsection 161(2) (that the amount determined under 161(2.2) is always less than or equal to the amount determined under 161(2)), let us compare the expression for subsection 161(2.2) in equation (B.2) above, with the expression for subsection 161(2) in equation (B.12). Recall that the expression for 161(2.2), as set out in equation (B.2), is as follows,

$$161(2.2) \;=\; \max\left[0,\ \sum_{i=1}^{12}\left(\sum_{j=1}^{i} q_j - \sum_{j=1}^{i} p_j\right)\bar{g}_i\right] \tag{B.13}$$

The sum of positive and negative amounts (as in equation (B.13)) must be less than the sum of the positive amounts alone (as in equation (B.12)). That is, summing amounts prior to taking the maximum of that sum and another amount, must be less than or equal to taking the maximum period by period and summing the results. Therefore, subsection 161(2.2) is less than or equal to subsection 161(2).

A more familiar way to present this result is that the contra-interest associated with overpayments under equation (B.13) can offset interest owing to the government from underpayments, whereas only positive amounts are permitted where offset interest is not permitted.

Proposition B2:

It is then demonstrated that the method of calculating instalment interest under Information Circular 81-11R, is in fact equivalent to the methods developed above (the amount of instalment interest determined in equations (B.1) and (B.2)). Exhibit III of Information

Circular 81-11R uses a running-balance method, where interest is calculated on the aggregate balance each period, to determine instalment interest under subsection 161(2.2). Defining B_i as the balance owing at payment date *i*, the following table demonstrates how the balance and interest is calculated in the Information Circular.

TABLE B.1

Method of Instalment Interest Calculation in Information Circular 81-11R

Payment date	Cumulative Balance	Interest Owing
1	$B_1 = (q_1 - p_1)$	$int_1 = B_1\left[\left(1+\frac{G_2}{365}\right)^{N_2} - 1\right]$
2	$B_2 = (q_2 - p_2) + B_1 + int_1$	$int_2 = B_2\left[\left(1+\frac{G_3}{365}\right)^{N_3} - 1\right]$
3	$B_3 = (q_3 - p_3) + B_2 + int_2$	$int_3 = B_3\left[\left(1+\frac{G_4}{365}\right)^{N_4} - 1\right]$
...	...	...
12	$B_{12} = (q_{12} - p_{12}) + B_{11} + int_{11}$	$int_{12} = B_{12}\left[\left(1+\frac{G_s}{365}\right)^{N_s} - 1\right]$
s	$B_s = B_{12} + int_{12}$	
Cumulative Balance and Interest	$B_s = B_{12} + int_{12}$	161(2.2) = $\max\left\{0, \sum_{i=1}^{12} int_i\right\}$

Proof:

To prove that the method used by Revenue Canada in Information Circular 81-11R is equivalent to 161(2.2), let us demonstrate that it is equivalent to the amount determined under equation (B.2) above; that is, that it equals,

$$U = \max\left[0, \sum_{i=1}^{12}\left(\sum_{j=1}^{i} q_j - \sum_{j=1}^{i} p_j\right)\overline{g_i}\right] \tag{B.14}$$

where

$$\overline{g_i} = \left[\left(1+\frac{G_{i+1}}{365}\right)^{N_{i+1}} - 1\right] * \prod_{k=i+2}^{13}\left(1+\frac{G_k}{365}\right)^{N_k}$$

Consider the two methods under the simplifying assumption that there are only two months instead of 12. Equation (B.14), for two months $i = 1,2$, may be rewritten,

for $i=1$,

$$(q_1 - p_1)\overline{g_1} = (q_1 - p_1)\left[\left(1+\frac{G_2}{365}\right)^{N_2} - 1\right] \cdot \left(1+\frac{G_s}{365}\right)^{N_s}$$

and for $i=2$,

$$\sum_{j=1}^{2}(q_j - p_j)\overline{g_i} = \left((q_1 - p_1) + (q_2 - q_2)\right)\left[\left(1+\frac{G_s}{365}\right)^{N_s} - 1\right]$$

and therefore, summing the amounts for these two months,

$$\sum_{i=1}^{2}\left(\sum_{j=1}^{i} q_j - \sum_{j=1}^{i} p_j\right)\bar{g}_i = (q_1-p_1)\left[\left(1+\frac{G_2}{365}\right)^{N_2} - 1\right] * \left(1+\frac{G_s}{365}\right)^{N_s}$$

$$+ \left((q_1-p_1)+(q_2-q_2)\right)\left[\left(1+\frac{G_s}{365}\right)^{N_s} - 1\right] \qquad \text{(B.15)}$$

To demonstrate equivalence, let us demonstrate that equations (B.15) equals the sum using the running balance method from Information Circular 81-11R. The amounts for two periods using the running balance method is int_1 and int_2 respectively. These amounts may be written,

$$int_1 = (q_1-p_1)\left[\left(1+\frac{G_2}{365}\right)^{N_2} - 1\right]$$

$$int_2 = B_2\left[\left(1+\frac{G_3}{365}\right)^{N_3} - 1\right]$$

$$= \left((q_2-p_2)+B_1+int_1\right)\left[\left(1+\frac{G_3}{365}\right)^{N_3} - 1\right]$$

$$= \left[(q_1-p_1) + (q_2-p_2) + (q_1-p_1)\left\{\left(1+\frac{G_2}{365}\right)^{N_2} - 1\right\}\right]\left[\left(1+\frac{G_3}{365}\right)^{N_3} - 1\right]$$

$$= \left[(q_1-p_1) + (q_2-p_2)\right]\left[\left(1+\frac{G_3}{365}\right)^{N_3} - 1\right] + (q_1-p_1)\left[\left(1+\frac{G_2}{365}\right)^{N_2} - 1\right] * \left(1+\frac{G_s}{365}\right)^{N_s}$$

$$- (q_1-p_1)\left[\left(1+\frac{G_s}{365}\right)^{N_s} - 1\right]$$

As the last term in int_2 equals $-int_1$, the sum of int_1 and int_2 is,

$$\sum_{i=1}^{2} int_i = (q_1 - p_1)\left[\left(1+\frac{G_2}{365}\right)^{N_2} - 1\right]\cdot\left(1+\frac{G_s}{365}\right)^{N_s} + \left((q_1-p_1)+(q_2-q_2)\right)\left[\left(1+\frac{G_s}{365}\right)^{N_s} - 1\right] \qquad \text{(B.17)}$$

which is equal to equation (B.15) above.

Q.E.D.

An Example Illustrating the Equivalence of the Three Methods

To demonstrate computationally the three equivalent formulations for U, the interest liability under subsection 161(2.2), let us replicate the example in Exhibit III of Information Circular 81-11R. For each of the three alternatives, the following assumptions are used:

Start of taxation year:	January 1, 1990
End of taxation year:	December 31, 1990
Remainder due date:	February 28, 1991
Instalment liability:	12 payments of $35,000

The prescribed rates of interest are those defined for each quarter of 1990. Payments are made throughout the year as defined in the tables below.

Let us start with the method used in Exhibit III as defined in Table B.1 above.

TABLE B.2

Calculation of Instalment Interest as Determined in Information Circular 81-11R

Date	q_i	p_i	N_{i+1}	G_{i+1}	Cumulative Balance	Interest
Jan. 15		35,000	16	15%	(35,000.00)	(230.85)
Jan. 31	35,000		28	15%	(230.85)	(2.67)
Feb. 28	35,000		15	15%	34,766.48	214.93
Mar. 15		35,000	16	15%	(18.59)	(.12)
Mar. 31	35,000		30	15%	34,981.29	433.86
Apr.30	35,000		31	15%	70,415.15	902.62
May 31	35,000		15	15%	106,317.80	657.27
June 15		35,000	15	15%	71,975.04	444.96
June 30	35,000		15	15%	107,420.00	708.49
July 15		35,000	16	16%	73,128.50	514.59
July 31	35,000		31	16%	108,643.10	1,486.10
Aug. 31	35,000		30	16%	145,129.20	1,920.73
Sept. 30	35,000		15	16%	147,049.90	969.88
Oct. 15		35,000	16	16%	113,019.80	795.29
Oct. 31	35,000	35,000	30	16%	113,815.10	1,506.30
Nov. 30	35,000	35,000	24	16%	115,321.40	1,219.38
Dec. 24		35,000	7	16%	81,540.77	250.54
Dec. 31	35,000	35,000	29	15%	81,791.31	980.40
Jan. 29		35,000	17	15%	47,771.71	334.85
Feb. 15		35,000	13	15%	13,106.56	70.19
			Total instalment interest			**$13,176.75**

In Table B.2 the cumulative balance at any date *i* reflects the aggregate amount owing by the corporation at that date. There are two components to this aggregate amount; the difference between the cumulative instalment liability and cumulative payments to that date, plus the amount of accumulated interest to that date.

In Table B.3 the second alternative for calculating instalment interest as defined in equation (B.1) above is examined:

$$U = \max\left[0, \sum_{i=1}^{12} (q_i - p_i) g_i\right].$$

where

$$g_i = \prod_{k=i+1}^{13} \left(1 + \frac{G_k}{365}\right)^{N_k} - 1$$

TABLE B.3

Calculation of Instalment Interest Using Equation (B.1)

Date	q_i	p_i	N_{i+1}	G_{i+1}	$q_i - p_i$	Interest $(q_i - p_i) g_i$
Jan. 15		35,000	16	15%	(35,000)	(6,681.21)
Jan. 31	35,000		28	15%	35,000	6,408.10
Feb. 28	35,000		15	15%	35,000	5,934.45
Mar. 15		35,000	16	15%	(35,000)	(5,682.94)
Mar. 31	35,000		30	15%	35,000	5,416.37
Apr.30	35,000		31	15%	35,000	4,921.24
May 31	35,000		15	15%	35,000	4,415.99
June 15		35,000	15	15%	(35,000)	(4,173.81)
June 30	35,000		15	15%	35,000	3,933.12
July 15		35,000	16	16%	(35,000)	(3,678.02)
July 31	35,000		31	16%	35,000	3,407.75
Aug. 31	35,000		30	16%	35,000	2,889.47
Sept. 30	35,000		15	16%	0	0
Oct. 15		35,000	16	16	(35,000)	(2,149.54)
Oct. 31	35,000	35,000	30	16%	0	0
Nov. 30	35,000	35,000	24	16%	0	0
Dec. 24		35,000	7	16%	(35,000)	(1,027.16)
Dec. 31	35,000	35,000	29	15%	0	0
Jan. 29		35,000	17	15%	(35,000)	(463.21)
Feb. 15		35,000	13	15%	(35,000)	(199.98)
Total instalment interest						**$13,176.75**

Finally, in Table B.4 the third alternative method of calculating instalment interest, as set out in equation (B.2), is examined:

$$U = \max\left[0, \sum_{i=1}^{12}\left(\sum_{j=1}^{i} q_j - \sum_{j=1}^{i} p_j\right)\overline{g}_i\right] .$$

where

$$\overline{g}_i = \left[\left(1 + \frac{G_{i+1}}{365}\right)^{N_{i+1}} - 1\right] * \prod_{k=i+2}^{13}\left(1 + \frac{G_k}{365}\right)^{N_k}$$

TABLE B.4

Calculation of Instalment Interest Using Equation (B.2)

Date	q_i	p_i	N_{i+1}	G_{i+1}	$\sum_{j=1}^{i} q_j - \sum_{j=1}^{i} p_j$	Interest $\left(\sum_{j=1}^{i} q_j - \sum_{j=1}^{i} p_j\right)\overline{g_i}$
Jan. 15		35,000	16	15%	(35,000)	(272.67)
Jan. 31	35,000		28	15%	0	0
Feb. 28	35,000		15	15%	35,000	251.10
Mar. 15		35,000	16	15%	0	0
Mar. 31	35,000		30	15%	35,000	494.32
Apr.30	35,000		31	15%	70,000	1,008.88
May 31	35,000		15	15%	105,000	725.36
June 15		35,000	15	15%	70,000	480.60
June 30	35,000		15	15%	105,000	764.07
July 15		35,000	16	16%	70,000	539.66
July 31	35,000		31	16%	105,000	1,552.33
Aug. 31	35,000		30	16%	140,000	1,976.42
Sept. 30	35,000		15	16%	140,000	978.51
Oct. 15		35,000	16	16	105,000	777.50
Oct. 31	35,000	35,000	30	16%	105,000	1,443.21
Nov. 30	35,000	35,000	24	16%	105,000	1,140.98
Dec. 24		35,000	7	16%	70,000	220.35
Dec. 31	35,000	35,000	29	15%	70,000	849.47
Jan. 29		35,000	17	15%	35,000	246.64
Feb. 15		35,000	13	15%	0	0
Total instalment interest						**$13,176.75**

APPENDIX C

THE PRESENT VALUE OF CASH FLOWS FORMULATION

An alternative objective function would be to minimize the present value, as of the start of a fiscal year, of all payments required by the Income Tax Act to be paid by the corporation to Revenue Canada in respect of that year (in this definition, negative payments are allowed as Revenue Canada may be required to make a refund to the corporation). Note that this alternative objective function may be equivalently stated as maximizing the present value of the cash flows.

The present value of all tax payments, denoted $t_{pv}(p;x)$, may be written,

$$\begin{aligned} t_{pv}(p;x) = & \sum_{i=1}^{12} \frac{p_i}{\prod_{j=0}^{i-1}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} \\ & + \frac{\max\left[0, \sum_{i=1}^{12} (q_i - p_i) g_i\right]}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} \\ & + \frac{.50 \cdot \max\left[0, \sum_{i=1}^{12} (q_i - p_i) g_i - \max\left(1000, .25 \sum_{i=1}^{12} q_i g_i\right)\right]}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} \\ & + \frac{\max\left[0, \sum_{i=1}^{12} (x - p_i)\right]}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} \\ & - \frac{\max\left[0, \sum_{i=1}^{12} (p_i - x)\right]}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}} \cdot \left(1+\frac{C_s}{365}\right)^{N_s}} - \frac{\max\left[0, \sum_{i=1}^{12} (p_i - x)\right] \cdot \left[\left(1+\frac{G_t}{365}\right)^{N_t} - 1\right]}{\left[\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}\right] \cdot \left(1+\frac{C_s}{365}\right)^{N_s}} \end{aligned} \tag{C.1}$$

Proposition C1:

The amount $t_{pv}(p;x)$ is a linear transformation of $l(p;x)$ where the coefficients are constants. That is,

$$t_{PV} = a_1 + a_2\, l(p;x) \qquad \text{(C.2)}$$

where,

$$a_1 = \frac{\sum_{i=1}^{12} q_i c_i + \sum_{i=1}^{12} x}{\prod_{j=0}^{12} \left(1 + \frac{C_{j+1}}{365}\right)^{N_{j+1}}}$$

and,

$$a_2 = \frac{1}{\prod_{j=0}^{12} \left(1 + \frac{C_{j+1}}{365}\right)^{N_{j+1}}}$$

Therefore, minimizing the expected value of $l(p;x)$ is equivalent to minimizing the expected value of $t_{pv}(p;x)$; *i.e.*, both problems produce the same optimal values of the decision variables, $p_1, p_2, \ldots, p_{12}$.

Proof:

Let us start by expanding the right hand side of equations (C.2) using equation (2.25) and (2.22) and rearranging terms such that the first five terms correspond to $a_2 \cdot l(p;x)$, and the final two terms correspond to a_1,

$$
\begin{aligned}
RHS = {} & \frac{\max\left[0, \sum_{i=1}^{12} (q_i - p_i) g_i\right]}{\prod_{j=0}^{12} \left(1 + \frac{C_{j+1}}{365}\right)^{N_{j+1}}} + \frac{\sum_{i=1}^{12} (p_i - q_i) c_i}{\prod_{j=0}^{12} \left(1 + \frac{C_{j+1}}{365}\right)^{N_{j+1}}} \\
& + \frac{.50 \cdot \max\left[0, \sum_{i=1}^{12} (q_i - p_i) g_i - \max\left(1000, .25 \sum_{i=1}^{12} q_i g_i\right)\right]}{\prod_{j=0}^{12} \left(1 + \frac{C_{j+1}}{365}\right)^{N_{j+1}}} \\
& + \frac{\max\left[0, \sum_{i=1}^{12} (p_i - x)\right] \cdot \left[\left(1 + \frac{C_s}{365}\right)^{N_s} - 1\right]}{\left[\prod_{j=0}^{12} \left(1 + \frac{C_{j+1}}{365}\right)^{N_{j+1}}\right] \cdot \left(1 + \frac{C_s}{365}\right)^{N_s}} - \frac{\max\left[0, \sum_{i=1}^{12} (p_i - x)\right] \cdot \left[\left(1 + \frac{G_z}{365}\right)^{N_t} - 1\right]}{\left[\prod_{j=0}^{12} \left(1 + \frac{C_{j+1}}{365}\right)^{N_{j+1}}\right] \cdot \left(1 + \frac{C_s}{365}\right)^{N_s}} \\
& + \frac{\sum_{i=1}^{12} q_i c_i}{\prod_{j=0}^{12} \left(1 + \frac{C_{j+1}}{365}\right)^{N_{j+1}}} + \frac{\sum_{i=1}^{12} x_i}{\prod_{j=0}^{12} \left(1 + \frac{C_{j+1}}{365}\right)^{N_{j+1}}}
\end{aligned}
\tag{C.3}
$$

Equation (C.3) may be rewritten, through expanding c_i using equation (2.14) and summing the second and sixth terms in that equation, as,

$$RHS = \frac{\max\left[0, \sum_{i=1}^{12} (q_i - p_i) g_i\right]}{\prod_{j=0}^{12}\left(1 + \frac{C_{j+1}}{365}\right)^{N_{j+1}}}$$

$$+ \frac{\sum_{i=1}^{12}\left[p_i\left(\prod_{j=i}^{12}\left(1 + \frac{C_{j+1}}{365}\right)^{N_{j+1}} - 1\right)\right]}{\prod_{j=0}^{12}\left(1 + \frac{C_{j+1}}{365}\right)^{N_{j+1}}}$$

$$+ \frac{.50 \cdot \max\left[0, \sum_{i=1}^{12}(q_i - p_i) g_i - \max\left(1000, .25 \sum_{i=1}^{12} q_i g_i\right)\right]}{\prod_{j=0}^{12}\left(1 + \frac{C_{j+1}}{365}\right)^{N_{j+1}}} \tag{C.4}$$

$$+ \frac{\max\left[0, \sum_{i=1}^{12}(p_i - x)\right] \cdot \left[\left(1 + \frac{C_s}{365}\right)^{N_s} - 1\right]}{\left[\prod_{j=0}^{12}\left(1 + \frac{C_{j+1}}{365}\right)^{N_{j+1}}\right] \cdot \left(1 + \frac{C_s}{365}\right)^{N_s}} - \frac{\max\left[0, \sum_{i=1}^{12}(p_i - x)\right] \cdot \left[\left(1 + \frac{G_z}{365}\right)^{N_z} - 1\right]}{\left[\prod_{j=0}^{12}\left(1 + \frac{C_{j+1}}{365}\right)^{N_{j+1}}\right] \cdot \left(1 + \frac{C_s}{365}\right)^{N_s}}$$

$$+ \frac{\sum_{i=1}^{12} x_i}{\prod_{j=0}^{12}\left(1 + \frac{C_{j+1}}{365}\right)^{N_{j+1}}}$$

This in turn equals (cancelling similar terms in the numerators and denominators and expanding),

$$RHS = \frac{\max\left[0, \sum_{i=1}^{12} (q_i - p_i) g_i\right]}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}}$$

$$+ \sum_{i=1}^{12} \frac{p_i}{\prod_{j=0}^{i-1}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} - \frac{\sum_{i=1}^{12} p_i}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}}$$

$$+ \frac{.50 \cdot \max\left[0, \sum_{i=1}^{12} (q_i - p_i) g_i - \max\left(1000, .25 \sum_{i=1}^{12} q_i g_i\right)\right]}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} \tag{C.5}$$

$$+ \frac{\max\left[0, \sum_{i=1}^{12} (p_i - x)\right]}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} - \frac{\max\left[0, \sum_{i=1}^{12} (p_i - x)\right]}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}} \cdot \left(1+\frac{C_s}{365}\right)^{N_s}}$$

$$- \frac{\max\left[0, \sum_{i=1}^{12} (p_i - x)\right] \cdot \left[\left(1+\frac{G_z}{365}\right)^{N_z} - 1\right]}{\left[\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}\right] \cdot \left(1+\frac{C_s}{365}\right)^{N_s}}$$

$$+ \frac{\sum_{i=1}^{12} x_i}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}}$$

The proof is complete if it can be shown that the difference between the left and right hand sides of equation (C.2) (which are respectively equation (C.1) and equation (C.5)), is zero. Since the first, third, fourth, sixth, and seventh terms in equation (C.5) are respectively identical to the second, first, third, fifth, and sixth terms in equation (C.1), the difference may be written,

$$\frac{\max\left[0,\sum_{i=1}^{12}(x-p_i)\right]}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} - \left[-\frac{\sum_{i=1}^{12}p_i}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} + \frac{\max\left[0,\sum_{i=1}^{12}(p_i-x)\right]}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} + \frac{\sum_{i=1}^{12}x_i}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}}\right] \tag{C.6}$$

where the first term is the remaining term from equation (C.1) and the amount in square brackets is the three remaining terms from equation (C.5).

Does this difference equal zero? Let us examine two cases:

Case 1:

If $\sum_{i=1}^{12}p_i > \sum_{i=1}^{12}x$, the difference is,

$$0 - \left[-\frac{\sum_{i=1}^{12}p_i}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} + \frac{\sum_{i=1}^{12}p_i-\sum_{i=1}^{12}x_i}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} + \frac{\sum_{i=1}^{12}x_i}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}}\right] = 0 \tag{C.7}$$

<u>Case 2:</u>

If $\sum_{i=1}^{12} p_i \leq \sum_{i=1}^{12} x$, the difference is,

$$\frac{\sum_{i=1}^{12} x_i - \sum_{i=1}^{12} p_i}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} - \left[- \frac{\sum_{i=1}^{12} p_i}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} + \frac{\sum_{i=1}^{12} x_i}{\prod_{j=0}^{12}\left(1+\frac{C_{j+1}}{365}\right)^{N_{j+1}}} \right] = 0 \qquad \text{(C.8)}$$

Therefore, the proposition is proven.

APPENDIX D

REWRITING THE PENALTY FUNCTION

The purpose of this appendix is to derive an equivalent expression for the penalty function without maximization and minimization operators.

1. Removing the Maximization Operators

Recall that the penalty payable under section 163.1 is;

$$Pen = .50 \cdot \max[0, (\min(X, B_1) - p)g - \max(1000, .25 \min(X, B_1)\, g)] \qquad (D.1)$$

To eliminate the right-hand maximization operator, it is convenient to rewrite equation (D.1) as follows:

$$Pen = \min(Pen_a, Pen_b) \qquad where \qquad (D.2)$$

$$Pen_a = .50 \cdot \max[0, (\min(X, B_1) - p)g - 1000] \qquad (D.3)$$

$$Pen_b = .50 \cdot \max[0, (.75 \cdot \min(X, B_1) - p)g] \qquad (D.4)$$

The maximization operator in equations (D.3) and (D.4) above may be removed as follows:

$$Pen_a = \begin{cases} .50 \cdot [(\min(X,B_1)-p)\,g - 1000] & \text{if} \quad (A) \quad \min(X,B_1) \geq p + \dfrac{1000}{g} \\ 0 & \text{otherwise} \end{cases} \qquad \textbf{(D.5)}$$

$$Pen_b = \begin{cases} .50 \cdot [.75 \cdot (\min(X,B_1)-p)\,g] & \text{if} \quad (B) \quad \min(X,B_1) \geq \dfrac{p}{.75} \\ 0 & \text{otherwise} \end{cases} \qquad \textbf{(D.6)}$$

Now let us return to a single stage definition of *Pen* through eliminating the functions Pen_a and Pen_b. Consider the three possible regions of *Pen*. The first region occurs where Pen_a is the minimum and is determined by its first argument. For this region, an additional condition (*C*), $min(X,x_{-12}) \leq 4000/g$ is derived through setting the first argument in Pen_a less than the first argument in Pen_b.[72] The second region occurs where Pen_b is the minimum and is determined by its first argument. For this second region, an additional condition (*D*), $\min(X,B_1) \geq 4000/g$ is derived through setting the first argument in Pen_b less than the first argument in Pen_a (the proof for condition *D* is the same as for *C* but with the inequality sign reversed). The third region occurs where either (*i*) Pen_a is the minimum and is determined by its second argument, or (*ii*) Pen_b is the minimum and is determined by its second argument.

[72]Setting the first argument less than the second argument,

$$.50\,[(\min(X,B_1)-p)\,g - 1000] \leq .50\,[.75\min(X,B_1)-p]\,g$$
$$\therefore \quad \min(X,B_1)\,g - .75\min(X,B_1)\,g \leq 1000$$
$$\therefore \quad .25\min(X,B_1) \leq 1000/g$$
$$\therefore \quad \min(X,B_1) \leq 4000/g$$

The penalty may therefore be rewritten,

$$Pen = \begin{cases} .50 * [(\min(X,B_1)-p)g - 1000] & \text{if } (A) \ \min(X,B_1) \geq p + 1000/g \\ & \text{and } (B) \ \min(X,B_1) \geq p/.75 \\ & \text{and } (C) \ \min(X,B_1) \leq 4000/g \\ .50 * [(.75\min(X,B_1)-p)g] & \text{if } (A) \ \min(X,B_1) \geq p + 1000/g \\ & \text{and } (B) \ \min(X,B_1) \geq p/.75 \\ & \text{and } (D) \ \min(X,B_1) \geq 4000/g \\ 0 & \text{otherwise} \end{cases} \tag{D.7}$$

2. Conditioning on p Relative to g

It is convenient to divide each of the first two branches of equation (D.7) into two by adding either the inequality $p \leq 3000/g$, or the inequality $p \geq 3000/g$:

$$Pen = \begin{cases} .50 * [(\min(X,B_1)-p)g - 1000] & \text{if } (A) \ \min(X,B_1) \geq p + 1000/g \\ & \text{and } (B) \ \min(X,B_1) \geq p/.75 \\ & \text{and } (C) \ \min(X,B_1) \leq 4000/g \\ & \text{and } (E) \ p \leq 3000/g \\ .50 * [(.75\min(X,B_1)-p)g] & \text{if } (A) \ \min(X,B_1) \geq p + 1000/g \\ & \text{and } (B) \ \min(X,B_1) \geq p/.75 \\ & \text{and } (D) \ \min(X,B_1) \geq 4000/g \\ & \text{and } (E) \ p \leq 3000/g \\ .50 * [(\min(X,B_1)-p)g - 1000] & \text{if } (A) \ \min(X,B_1) \geq p + 1000/g \\ & \text{and } (B) \ \min(X,B_1) \geq p/.75 \\ & \text{and } (C) \ \min(X,B_1) \leq 4000/g \\ & \text{and } (F) \ p \geq 3000/g \\ .50 * [(.75\min(X,B_1)-p)g] & \text{if } (A) \ \min(X,B_1) \geq \min p + 1000/g \\ & \text{and } (B) \ \min(X,B_1) \geq p/.75 \\ & \text{and } (D) \ \min(X,B_1) \geq 4000/g \\ & \text{and } (F) \ p \geq 3000/g \\ 0 & \text{otherwise} \end{cases} \tag{D.8}$$

The following lemmas demonstrate that each of the first four branches of *Pen* as defined above contain redundant inequalities:

Lemma D1: If (*A*) and (*E*), then (*B*)

Proof: Multiply (*A*) by 3 and (*E*) by -1. Adding the resulting inequalities gives $3*\min(X,B_1) \geq 4p$. Dividing by 3 gives $\min(X,B_1) \geq p/.75$ which is (*B*).

Lemma D2: If (*D*) and (*E*), then (*A*)

Proof: Multiplying (*E*) by -1 and adding the result to (*D*), gives $\min(X,B_1) \geq p+1000/g$ which is (*A*).

Lemma D3: If (*B*) and (*F*), then (*A*)

Proof: Multiply (*F*) by 3 and add the result to (*B*), gives $\min(X,B_1) \geq p + 1000/g$ which is (*A*).

Lemma D4: If (*B*) and (*F*), then (*D*)

Proof: Multiply (*F*) by 4/3 and add the result to (*B*) to obtain $\min(X,B_1) \geq 4000/g$ which is (*D*).

Removing the redundant inequalities, equation (D.8) may be rewritten:

$$Pen = \begin{cases} .50 * [(\min(X,B_1)-p)g - 1000] & \text{if } (A) \ \min(X,B_1) \geq p + 1000/g \\ & \text{and } (C) \ \min(X,B_1) \leq 4000/g \\ & \text{and } (E) \ p \leq 3000/g \\ .50 * [(.75\min(X,B_1)-p)g] & \text{if } (D) \ \min(X,B_1) \geq 4000/g \\ & \text{and } (E) \ p \leq 3000/g \\ .50 * [(\min(X,B_1)-p)g - 1000] & \text{if } (B) \ \min(X,B_1) \geq p/.75 \\ & \text{and } (C) \ \min(X,B_1) \leq 4000/g \\ & \text{and } (F) \ p \geq 3000/g \\ .50 * [(.75\min(X,B_1)-p)g] & \text{if } (B) \ \min(X,B_1) \geq p/.75 \\ & \text{and } (F) \ p \geq 3000/g \\ 0 & \text{otherwise} \end{cases} \qquad (D.9)$$

The third branch of the function can be eliminated as the region over which it is valid (the conjunction of inequalities (*B*), (*C*), and (*F*)) is a subset of the region over which the fourth branch of the function is valid (the conjunction of inequalities (*B*) and (*F*)).[73] After eliminating the redundant region, the function becomes:

$$Pen = \begin{cases} .50 * [(\min(X,B_1)-p)g - 1000] & \text{if } (A) \ \min(X,B_1) \geq p + 1000/g \\ & \text{and } (C) \ \min(X,B_1) \leq 4000/g \\ & \text{and } (E) \ p \leq 3000/g \\ .50 * [(.75\min(X,B_1)-p)g] & \text{if } (D) \ \min(X,B_1) \geq 4000/g \\ & \text{and } (E) \ p \leq 3000/g \\ .50 * [(.75\min(X,B_1)-p)g] & \text{if } (B) \ \min(X,B_1) \geq p/.75 \\ & \text{and } (F) \ p \geq 3000/g \\ 0 & \text{otherwise} \end{cases} \qquad (D.10)$$

---

[73]Another way to view this situation is that (*B*), (*C*) and (*F*) together imply $p = 3000/g$. At that value of p, the function value is the same for both the third and fourth branches of the function.

or (rearranging),

$$
Pen = \begin{cases}
.50 * [(.75\min(X,B_1)-p)g] & \text{if } (D)\ \min(X,B_1) \geq 4000/g \\
 & \text{and } (E)\ p \leq 3000/g \\
.50 * [(\min(X,B_1)-p)g - 1000] & \text{if } (A)\ \min(X,B_1) \geq p + 1000/g \\
 & \text{and } (C)\ \min(X,B_1) \leq 4000/g \\
 & \text{and } (E)\ p \leq 3000/g \\
.50 * [(.75\min(X,B_1)-p)g] & \text{if } (B)\ \min(X,B_1) \geq p/.75 \\
 & \text{and } (F)\ p \geq 3000/g \\
0 & \text{otherwise}
\end{cases} \quad \text{(D.11)}
$$

3. Removing the Minimization Operators

Consider the inequalities $X \geq B_1$ and $X \leq B_1$. If the former holds, $\min(X,B_1) = B_1$, whereas if the latter holds, $\min(X,B_1) = X$. Therefore, the minimization operator can be eliminated by dividing each of the branches above into two by adding either of the above two inequalities:

$$Pen = \begin{cases} .50 * [(.75B_1 - p)g] & \text{if } X \geq B_1 \geq 4000/g \geq \dfrac{p}{.75} \\ & \text{and } p \leq 3000/g \\ .50 * [(.75X - p)g] & \text{if } B_1 \geq X \geq 4000/g \geq \dfrac{p}{.75} \\ & \text{and } p \leq 3000/g \\ .50 * [(B_1 - p)g - 1000] & \text{if } \{X, 4000/g\} \geq B_1 \geq p + 1000/g \\ & \text{and } p \leq 3000/g \\ .50 * [(X - p)g - 1000] & \text{if } \{B_1, 4000/g\} \geq X \geq p + 1000/g \\ & \text{and } p \leq 3000/g \\ .50 * [(.75B_1 - p)g] & \text{if } X \geq B_1 \geq p/.75 \\ & \text{and } p \geq 3000/g \\ .50 * [(.75X - p)g] & \text{if } B_1 \geq X \geq p/.75 \\ & \text{and } p \geq 3000/g \\ 0 & \text{otherwise} \end{cases} \qquad \text{(D.12)}$$

4. Putting *Pen* into a More Useful Form

The above definition of *Pen* is not yet in a form suitable for integration over X (which is needed to define $L(p)$). The problem is that one set of inequalities defining the branches describes the relationship among p and the parameters involved in the penalty (g and B_1) while another set of inequalities put upper and lower bounds on X (the variable to be integrated over). For the purposes of performing integrations over X, those two sets of inequalities need to be separated. Let us create a definition of *Pen* which branches initially on the first set of inequalities and subsequently branches on the second set of inequalities.

The penalty may be rewritten,

$$Pen = \begin{cases} Pen_1 & \text{if } p \le \frac{3000}{g} \text{ and } B_1 \ge \frac{4000}{g} \\ Pen_2 & \text{if } p \le \frac{3000}{g} \text{ and } \frac{4000}{g} \ge B_1 \ge p + \frac{1000}{g} \\ Pen_3 & \text{if } p \le \frac{3000}{g} \text{ and } p + \frac{1000}{g} \ge B_1 \\ Pen_4 & \text{if } p \ge \frac{3000}{g} \text{ and } B_1 \ge \frac{p}{.75} \\ Pen_5 & \text{if } p \ge \frac{3000}{g} \text{ and } B_1 \le \frac{p}{.75} \end{cases} \qquad \text{(D.13)}$$

where,

$$Pen_1 = \begin{cases} .50 \cdot [(.75\,B_1 - p)g] & if \quad X \geq B_1 \\ .50 \cdot [(.75\,X - p)g] & if \quad B_1 \geq X \geq \dfrac{4000}{g} \\ .50 \cdot [(X - p)g - 1000] & if \quad \dfrac{4000}{g} \geq X \geq p + \dfrac{1000}{g} \\ 0 & otherwise \end{cases} \tag{D.14}$$

$$Pen_2 = \begin{cases} .50 \cdot [(B_1 - p)g - 1000] & if \quad X \geq \dfrac{4000}{g} \\ .50 \cdot [(B_1 - p)g - 1000] & if \quad \dfrac{4000}{g} \geq X \geq B_1 \\ .50 \cdot [(X - p)g - 1000] & if \quad B_1 \geq X \geq p + \dfrac{1000}{g} \\ 0 & otherwise \end{cases} \tag{D.15}$$

$$Pen_3 = 0 \tag{D.16}$$

$$Pen_4 = \begin{cases} .50 \cdot [(.75\,B_1 - p)g] & if \quad X \geq B_1 \\ .50 \cdot [(.75\,X - p)g] & if \quad B_1 \geq X \geq \dfrac{p}{.75} \\ 0 & otherwise \end{cases} \tag{D.17}$$

$$Pen_5 = 0 \tag{D.18}$$

5. Combining to form a Revised Expression for the Corporation's Loss

The penalty loss, *Pen*, may be added to the interest loss from equation (3.14) of chapter 3 to form an equivalent expression for corporation's loss. Recall that the loss without penalty is, from equation (3.14),

$$l(p;X) = \begin{cases} (g-c)\cdot(B_1-p) & for\ \ X \geq B_1 \\ (g-c)\cdot(X-p) & for\ \ B_1 \geq X \geq p \\ c\cdot(p-X) & for\ \ p \geq X \geq 0 \end{cases} \qquad \textbf{(D.19)}$$

The partitions utilized in the penalty structure will continue to be utilized. The loss with penalty may be written;

$$l(p;X) = \begin{cases} e_1 & if\ \ p \leq \frac{3000}{g} & and\ \ B_1 \geq \frac{4000}{g} \\ e_2 & if\ \ p \leq \frac{3000}{g} & and\ \ \frac{4000}{g} \geq B_1 \geq p + \frac{1000}{g} \\ e_3 & if\ \ p \leq \frac{3000}{g} & and\ \ p + \frac{1000}{g} \geq B_1 \\ e_4 & if\ \ p \geq \frac{3000}{g} & and\ \ B_1 \geq \frac{p}{.75} \\ e_5 & if\ \ p \geq \frac{3000}{g} & and\ \ B_1 \leq \frac{p}{.75} \end{cases} \qquad \textbf{(D.20)}$$

where

$$e_1 = \begin{cases} .50 \cdot [(.75\,B_1 - p)g] + (g-c)\cdot(B_1 - p) \\ \qquad\qquad if \quad X \geq B_1 \\ .50 \cdot [(.75\,X - p)g] + (g-c)\cdot(X-p) \\ \qquad\qquad if \quad B_1 \geq X \geq \dfrac{4000}{g} \\ .50 \cdot [(X-p)g - 1000] + (g-c)\cdot(X-p) \\ \qquad\qquad if \quad \dfrac{4000}{g} \geq X \geq p + \dfrac{1000}{g} \\ (g-c)\cdot(X-p) \quad if \quad p + \dfrac{1000}{g} \geq X \geq p \\ c \cdot (p - X) \quad if \quad p \geq X \end{cases} \tag{D.21}$$

$$e_2 = \begin{cases} .50\, codt\, [(B_1 - p)g - 1000] + (g-c)\cdot(B_1 - p) \\ \qquad\qquad if \quad X \geq \dfrac{4000}{g} \\ .50 \cdot [(B_1 - p)g - 1000] + (g-c)\cdot(B_1 - p) \\ \qquad\qquad if \quad \dfrac{4000}{g} \geq X \geq B_1 \\ .50 \cdot [(X-p)g - 1000] + (g-c)\cdot(X-p) \\ \qquad\qquad if \quad B_1 \geq X \geq p + \dfrac{1000}{g} \\ (g-c)\cdot(X-p) \quad if \quad p + \dfrac{1000}{g} \geq X \geq p \\ c \cdot (p - X) \quad if \quad p \geq X \end{cases} \tag{D.22}$$

$$e_3 = \begin{cases} (g-c)\cdot(B_1 - p) \quad if \quad X \geq B_1 \\ (g-c)\cdot(X-p) \quad if \quad B_1 \geq X \geq p \\ c \cdot (p - X) \quad if \quad p \geq X \end{cases} \tag{D.23}$$

$$e_4 = \begin{cases} .50 \cdot [(.75 B_1 - p) g] + (g - c) \cdot (B_1 - p) & \\ \qquad\qquad if \quad X \geq B_1 & \\ .50 \cdot [(.75 X - p) g] + (g - c) \cdot (X - p) & \\ \qquad\qquad if \quad B_1 \geq X \geq \dfrac{p}{.75} & \\ (g - c) \cdot (X - p) \qquad if \quad \dfrac{p}{.75} \geq X \geq p & \\ c \cdot (p - X) \qquad if \quad p \geq X & \end{cases} \tag{D.24}$$

$$e_5 = \begin{cases} (g - c) \cdot (B_1 - p) & if \quad X \geq B_1 \\ (g - c) \cdot (X - p) & if \quad B_1 \geq X \geq p \\ c \cdot (p - X) & if \quad p \geq X \end{cases} \tag{D.25}$$

These expressions can be simplified through combining regions to produce equations (3.29) through (3.32) in chapter 3.

APPENDIX E

DERIVING THE OPTIMAL CONDITIONS

The expected loss function with penalty, from equations (3.38) to (3.32) of chapter 3, and using the definition in equation (3.10), is

$$L(p \mid p \le B_1) = \begin{cases} L_1 & \text{if } p \le \dfrac{3000}{g} \text{ and } B_1 \ge \dfrac{4000}{g} \\ L_2 & \text{if } p \le \dfrac{3000}{g} \text{ and } \dfrac{4000}{g} \ge B_1 \ge p + \dfrac{1000}{g} \\ L_3 & \text{if } p \le \dfrac{3000}{g} \text{ and } p + \dfrac{1000}{g} \ge B_1 \\ L_4 & \text{if } p \ge \dfrac{3000}{g} \text{ and } B_1 \ge \dfrac{p}{.75} \\ L_5 & \text{if } p \ge \dfrac{3000}{g} \text{ and } B_1 \le \dfrac{p}{.75} \end{cases} \tag{E.1}$$

where

$$\begin{aligned} L_1 = & \int_{B_1}^{\infty} [(1.5g - c)\cdot(B_1 - p) - .125\, g\, B_1]\, f(X)\, dX \\ & + \int_{4000/g}^{B_1} [(1.5g - c)\cdot(X - p) - .125\, g\, X]\, f(X)\, dX \\ & + \int_{p+1000/g}^{4000/g} [(1.5g - c)\cdot(X - p) - 500]\, f(X)\, dX \\ & + \int_{p}^{p+1000/g} (g - c)\cdot(X - p)\, f(X)\, dX + \int_{0}^{p} (c - g)\cdot(p - X)\, f(X)\, dX \end{aligned} \tag{E.2}$$

$$L_2 = \int_{B_1}^{\infty} [(1.5\,g-c)\cdot(B_1-p) - 500]\, f(X)\, dX$$

$$+ \int_{p+1000/g}^{B_1} [(1.5\,g-c)\cdot(X-p) - 500]\, f(X)\, dX \qquad \text{(E.3)}$$

$$+ \int_{p}^{p+1000/g} (g-c)\cdot(X-p)\, f(X)\, dX + \int_{0}^{p} (c-g)\cdot(p-X)\, f(X)\, dX$$

$$L_3 = \int_{B_1}^{\infty} (g-c)\cdot(B_1-p)\, f(X)\, dX + \int_{p}^{B_1} (g-c)\cdot(X-p)\, f(X)\, dX \qquad \text{(E.4)}$$

$$+ \int_{0}^{p} c\cdot(p-X)\, f(X)\, dX$$

$$L_4 = \int_{B_1}^{\infty} [(1.5\,g-c)\cdot(B_1-p) - .125\,g\,B_1]\, f(X)\, dX$$

$$+ \int_{p/.75}^{B_1} [(1.5\,g-c)\cdot(X-p) - .125\,g\,X]\, f(X)\, dX \qquad \text{(E.5)}$$

$$+ \int_{p}^{p/.75} (g-c)\cdot(X-p)\, f(X)\, dX + \int_{0}^{p} c\cdot(p-X)\, f(X)\, dX$$

$$L_5 = \int_{B_1}^{\infty} (g-c)\cdot(B_1-p)\, f(X)\, dX + \int_{p}^{B_1} (g-c)\cdot(X-p)\, f(X)\, dX \qquad \text{(E.6)}$$

$$+ \int_{0}^{p} c\cdot(p-X)\, f(X)\, dX$$

Choosing p to minimize the expected loss L, the rule for differentiating an integral with respect to a parameter is utilized. The resulting derivatives are:

$$\frac{\partial L_1}{\partial p} = -(1.5g-c)\int_{p+1000/g}^{\infty} f(X)\,dX - (g-c)\int_{p}^{p+1000/g} f(X)\,dX + c\int_{0}^{p} f(X)\,dX \tag{E.7}$$

$$\frac{\partial L_2}{\partial p} = -(1.5g-c)\int_{p+1000/g}^{\infty} f(X)\,dX - (g-c)\int_{p}^{p+1000/g} f(X)\,dX + c\int_{0}^{p} f(X)\,dX \tag{E.8}$$

$$\frac{\partial L_3}{\partial p} = -(g-c)\int_{p}^{\infty} f(X)\,dX + c\int_{0}^{p} f(X)\,dX \tag{E.9}$$

$$\frac{\partial L_4}{\partial p} = -(1.5g-c)\int_{p/.75}^{\infty} f(X)\,dX - (g-c)\int_{p}^{p/.75} f(X)\,dX + c\int_{0}^{p} f(X)\,dX \tag{E.10}$$

$$\frac{\partial L_5}{\partial p} = -(g-c)\int_{p}^{\infty} f(X)\,dX + c\int_{0}^{p} f(X)\,dX \tag{E.11}$$

It is convenient to examine the problem separately for each of two sets of parameter values: first, for $B_1 \leq 4000/g$; and second, for $B_1 \geq 4000/g$. For a subset of the first set of parameter values, $B_1 \leq 1000/g$, the optimal solution is identical to the pre-1990 (no penalty) solution (equations (3.23), (3.24), and (3.25)). Intuitively, if $B_1 \leq 1000/g$, it is impossible to have a penalty. As these solutions were fully derived and discussed earlier, they will not be further examined in this appendix.

Case of $B_1 \leq 4000/g$

The optimization problem to be solved is,

$$\min_p L(p) \qquad \textit{subject to} \qquad 0 \leq p \leq B_1 . \qquad \text{(E.12)}$$

Since the expected loss function is not continuously differentiable, special analytic tools are required to solve the above problem. The appropriate tools, as developed by Macnaughton [1993], are the Diewert [1981] conditions. These conditions, applied to equation (E.12), are necessary conditions for an optimum. The conditions for this problem are,

$$(i)\quad p^* = 0 \qquad if \quad \frac{\partial L_2(0)}{\partial p} \geq 0$$

$$(ii)\quad 0 < p^* < B_1 - \frac{1000}{g} \qquad if \quad \frac{\partial L_2(p^*)}{\partial p} = 0$$

$$(iii)\quad p^* = B_1 - \frac{1000}{g} \qquad if \quad \frac{\partial L_2(p^*)}{\partial p} \leq 0 \leq \frac{\partial L_3(p^*)}{\partial p} \qquad \text{(E.13)}$$

$$(iv)\quad B_1 - \frac{1000}{g} < p^* < \frac{3000}{g} \qquad if \quad \frac{\partial L_3(p^*)}{\partial p} = 0$$

$$(v)\quad p^* = \frac{3000}{g} \qquad if \quad \frac{\partial L_3(p^*)}{\partial p} \leq 0 \leq \frac{\partial L_5(p^*)}{\partial p}$$

$$(vi)\quad p^* > \frac{3000}{g} \qquad if \quad \frac{\partial L_5(p^*)}{\partial p} = 0$$

$$(vii)\quad p^* = B_1 \qquad if \quad \frac{\partial L_5(p^*)}{\partial p} \leq 0$$

Since $\partial L_3/\partial p = \partial L_5/\partial p$, conditions *iv* through *vii* can be combined as:

$$p^* > B_1 - \frac{1000}{g} \qquad if \quad \frac{\partial L_{3,5}(p^*)}{\partial p} = 0$$

Intuition for these conditions may be given as follows. Consider first where the optimal solution occurs at the kink: $p^* = B_1 - 1000/g$. The Diewert [1981] conditions state that at an optimal point, the one-sided directional derivatives in any feasible direction must be non-positive. In this case, p^* can be either increased or decreased, *i.e.* the feasible directions are +1 and -1. Therefore, the Diewert conditions are:

$$L'(x_p;1) \geq 0 \quad and \quad L'(x_p;-1) \geq 0 \tag{E.14}$$

where L' are one sided derivatives in the direction 1 and -1 respectively. In essence, these Diewert conditions state that at an optimum, the derivative in any feasible direction must point up or be flat. We want to substitute ordinary derivatives in place of these one sided directional derivatives. This may be done by modifying a theorem which states a two sided directional derivative is equal to the cross-product of the direction and the gradient vector. The required modification to one-sided directional derivatives is Macnaughton's [1993] Lemma 2. For this case it states:

$$L'(p^*;1) = 1^T \frac{\partial L_3}{\partial p}(p^*) = \frac{\partial L_3}{\partial p}(p^*)$$

$$L'(p^*;-1) = -1^T \frac{\partial L_2}{\partial p}(p^*) = -\frac{\partial L_2}{\partial p}(p^*)$$

Substituting these values in equation (E.14) gives

$$\frac{\partial L_3}{\partial p}(p^*) \geq 0 \qquad \textit{and} \quad -\frac{\partial L_2}{\partial p}(p^*) \geq 0$$

which implies that

$$\frac{\partial L_2}{\partial p}(p^*) \leq 0 \leq \frac{\partial L_3}{\partial p}(p^*)$$

As a second example, consider $p^* = 0$. There is only a single feasible direction which is to increase p, so the Diewert [1981] conditions are

$$L'(p^*;1) \geq 0 \qquad \textit{and}$$
$$L'(p^*;1) = \frac{\partial L_2}{\partial p}(p^*)$$

For a third example, let us look at an interior optimum which is not at a kink. The Diewert [1981] conditions are

$$L'(p^*;1) \geq 0 \qquad \textit{and} \quad L'(p^*;-1) \geq 0 .$$

Since

$$L'(p^*;1) = \frac{\partial L_2}{\partial p}(p^*) \qquad \textit{and} \quad L'(p^*;-1) = -\frac{\partial L_2}{\partial p}(p^*)$$

it follows that

$$\frac{\partial L_2}{\partial x}(p^*) \geq 0 \qquad and \quad -\frac{\partial L_2}{\partial x}(p^*) \geq 0$$

$$\therefore \quad \frac{\partial L_2}{\partial x}(p^*) = 0$$

Note that the conditions in equation (E.13) did not utilize L_1 and L_4. The branches of the function having values L_1 and L_4 became subsets of regions L_2 and L_5 respectively. We therefore wanted to determine the nature of the function moving from L_2 to L_3 to L_5.

Performing the required differentiations and substituting into equation (E.13) above, the necessary and sufficient conditions are:

$$(i)\quad p^* = 0 \qquad if \quad -(1.5g-c)\int_{1000/g}^{\infty} f(X)\,dX - (g-c)\int_{0}^{1000/g} f(X)\,dX \;\geq\; 0$$

$$(ii)\quad 0 < p^* < B_1 - \frac{1000}{g} \qquad if \quad -(1.5g-c)\int_{p^*+1000/g}^{\infty} f(X)\,dX - (g-c)\int_{p^*}^{p^*+1000/g} f(X)\,dX + c\int_{0}^{p^*} f(X)\,dX \;=\; 0$$

$$(iii)\quad p^* = B_1 - \frac{1000}{g} \qquad if \quad -(1.5g-c)\int_{B_1}^{\infty} f(X)\,dX - (g-c)\int_{B_1-1000/g}^{B_1} f(X)\,dX + (c-g)\int_{0}^{B_1-1000/g} f(X)\,dX \;\leq\; 0 \;\leq\; -(g-c)\int_{B_1-1000/g}^{\infty} f(X)\,dX + c\int_{0}^{B_1-1000/g} f(X)\,dX \qquad \text{(E.15)}$$

$$(iv)\quad p^* > B_1 - \frac{1000}{g} \qquad if \quad -(g-c)\int_{p^*}^{\infty} f(X)\,dX + c\int_{0}^{p^*} f(X)\,dX \;=\; 0$$

$$(v)\quad p^* = B_1 \qquad if \quad (g-c)\int_{p^*}^{\infty} f(X)\,dX - c\int_{0}^{p^*} f(X)\,dX \;\geq\; 0$$

Note that a single kink occurs at $p = B_1 - 1000/g$. At that value of p the left hand derivative is less than the right hand derivative. With some additional manipulation, conditions (*i*) though (*v*) from equation (E.15) can be shown to be equivalent to (*i*) though (*v*) in equation (3.34) of chapter 3.

Case of $B_1 \geq 4000/g$

By similar reasoning, the Diewert conditions for an optimum for $B_1 \geq 4000/g$ may be written:

$$(i) \quad p^* = 0 \qquad if \quad \frac{\partial L_1(0)}{\partial p} \geq 0$$

$$(ii) \quad 0 < p^* < \frac{3000}{g} \qquad if \quad \frac{\partial L_1(p^*)}{\partial p} = 0$$

$$(iii) \quad p^* = \frac{3000}{g} \qquad if \quad \frac{\partial L_1(p^*)}{\partial p} \leq 0 \leq \frac{\partial L_4(p^*)}{\partial p}$$

$$(iv) \quad \frac{3000}{g} < p^* < .75\,B_1 \qquad if \quad \frac{\partial L_4(p^*)}{\partial p} = 0 \tag{E.16}$$

$$(v) \quad p^* = .75\,B_1 \qquad if \quad \frac{\partial L_4(p^*)}{\partial p} \leq 0 \leq \frac{\partial L_5(p^*)}{\partial p}$$

$$(iv) \quad p^* > .75\,B_1 \qquad if \quad \frac{\partial L_5(p^*)}{\partial p} = 0$$

$$(vii) \quad p^* = B_1 \qquad if \quad \frac{\partial L_5(p^*)}{\partial p} \leq 0$$

Again performing the required differentiations and substituting above, the necessary conditions are:

$$(i) \quad p^* = 0 \qquad if \quad -(1.5g-c)\int_{1000/g}^{\infty} f(X)\,dX \; - (g-c)\int_{0}^{1000/g} f(X)\,dX \;\geq\; 0$$

$$(ii) \quad 0 < p^* < \frac{3000}{g} \qquad if \quad -(1.5g-c)\int_{p^*+1000/g}^{\infty} f(X)\,dX \; - (g-c)\int_{p^*}^{p^*+1000/g} f(X)\,dX$$

$$+ \; c\int_{0}^{p} f(X)\,dX \; = \; 0$$

$$(iii) \quad p^* = \frac{3000}{g} \qquad if \quad -(1.5g-c)\int_{4000/g}^{\infty} f(X)\,dX \; - (g-c)\int_{3000/g}^{4000/g} f(X)\,dX$$

$$+ \; c\int_{0}^{3000/g} f(X)\,dX \quad \leq \; 0 \; \leq$$

$$-(1.5g-c)\int_{4000/g}^{\infty} f(X)\,dX \; - (g-c)\int_{3000/g}^{4000/g} f(X)\,dX \qquad \text{(E.17)}$$

$$+ \; c\int_{0}^{3000/g} f(X)\,dX$$

$$(iv) \quad \frac{3000}{g} < p^* < .75\,B_1 \qquad if \quad -(1.5g-c)\int_{p^*/.75}^{\infty} f(X)\,dX \; - (g-c)\int_{p^*}^{p^*/.75} f(X)\,dX$$

$$+ \; c\int_{0}^{p^*} f(X)\,dX \; = \; 0$$

$$(v) \quad p^{*} = .75\,B_1 \quad \text{if} \quad -(1.5\,g-c)\int_{B_a}^{\infty} f(X)\,dX \; - (g-c)\int_{.75B_1}^{B_1} f(X)\,dX$$

$$+\, c\int_{0}^{.75B_1} f(X)\,dX \quad \leq \; 0 \; \leq$$

$$-(g-c)\int_{.75B_1}^{\infty} f(X)\,dX \; + \; c\int_{0}^{.75B_1} f(X)\,dX$$

$$(iv) \quad p^{*} > .75\,B_1 \quad \text{if} \quad -(g-c)\int_{p^{*}}^{\infty} f(X)\,dX \; + \; c\int_{0}^{p^{*}} f(X)\,dX \; = \; 0$$

$$(vii) \quad p^{*} = B_1 \quad \text{if} \quad (g-c)\int_{p^{*}}^{\infty} f(X)\,dX - c\int_{0}^{p^{*}} f(X)\,dX \; \geq \; 0$$

Note that there is not a kink at $p^{*} = 3000/g$ as the derivative from the right and left are equal. Therefore, for $B_1 \geq 4000/g$ the only kink occurs at $p = .75B_1$ where the left hand derivative is less than the right hand derivative. With some additional manipulation, conditions (*i*) though (*vii*) from equation (E.17) can be shown to be equivalent to (*i*) though (*v*) in equation (3.35) of chapter 3.

APPENDIX F

CHAPTER 4 PROOFS

The ordering of the proofs in this appendix are as follows: convexity of $L(p)$; Lemmas 4.1 and 4.2; Proposition 4.1; Corollary 4.1; Lemma 4.3; proof of necessity for Proposition 4.3; Lemmas 4.4 through 4.7; and Proposition 4.4. The ordering is such that lemmas required in the proof of each proposition are presented prior to that proposition. Note that Lemmas 4.3 through 4.7 are not referred to in the text of chapter 4, but are used in the proofs of propositions 4.3 and 4.4.

Proof that $L(p)$ is Convex:

This result is used in section 4.3. By definition, $L(p)$ is convex if, for any two points p_a and p_b,

$$L(\lambda p_a + (1-\lambda) p_b) \leq \lambda L(p_a) + (1-\lambda) L(p_b) \tag{F.1}$$

where $0 \leq \lambda \leq 1$.

From equation (4.6) of chapter 4,

$$L(p) = \sum_{\omega=1}^{\Omega} Prob_{\omega} \sum_{n=1}^{4} f_{\omega n}(p)$$

which shows that $L(p)$ is a linear combination of the functions $f_{\omega i}$. Since any linear combination of convex functions is also convex, it suffices to show that each $f_{\omega i}$ is convex.

Consider $f_{\omega 1}(p)$, which from equations (4.7) and (4.11) can be written as $f_{\omega 1}(p) = \max\left(0, (q^{\omega} - p^{\omega})g\right)$, where p^{ω} is the 12-element sub-vector of p relating to a particular ω and q^{ω} and g are vectors. It is necessary to prove that,

$$f_{\omega 1}\left(\lambda p_a^{\omega} + (1-\lambda)p_b^{\omega}\right) \le f_{\omega 1}\left(p_a^{\omega}\right) + (1-\lambda)f_{\omega 1}\left(p_b^{\omega}\right) \qquad \text{(F.2)}$$

where p_a^{ω} and p_b^{ω} are any two 12 element sub-vectors of p relating to a particular ω. Substituting equations (4.7) and (4.11) into equation (F.1) above, the left hand side is,

$$f_{\omega 1}\left(\lambda p_a^{\omega} + (1-\lambda)p_b^{\omega}\right) = \begin{cases} \lambda(q^{\omega} - p_a^{\omega})g + (1-\lambda)(q^{\omega} - p_b^{\omega})g & \\ \qquad\qquad \text{if } \lambda(q^{\omega} - p_a^{\omega})g + (1-\lambda)(q^{\omega} - p_b^{\omega})g \ge 0 & \\ 0 \qquad\qquad \text{otherwise} & \end{cases} \qquad \text{(F.3)}$$

and the right hand side is,

$$\lambda f_{\omega 1}\left(p_a^{\omega}\right) + (1-\lambda)f_{\omega 1}\left(p_b^{\omega}\right)$$

$$= \begin{cases} \lambda(q^{\omega} - p_a^{\omega})g + (1-\lambda)(q^{\omega} - p_b^{\omega})g & \text{if } (q^{\omega} - p_a^{\omega})g \ge 0, \text{ and } (q^{\omega} - p_b^{\omega})g \ge 0 \quad \underline{B1} \\ \lambda(q^{\omega} - p_a^{\omega})g & \text{if } (q^{\omega} - p_a^{\omega})g \ge 0 \ge (q^{\omega} - p_b^{\omega})g \quad \underline{B2} \\ (1-\lambda)(q^{\omega} - p_b^{\omega})g & \text{if } (q^{\omega} - p_b^{\omega})g \ge 0 \ge (q^{\omega} - p_a^{\omega})g \quad \underline{B3} \\ 0 & \text{if } (q^{\omega} - p^{\omega})g \le 0, \text{ and } (q^{\omega} - p_b^{\omega})g \le 0 \quad \underline{B4} \end{cases}$$

(F.4)

Comparing equations (F.3) and (F.4) branch by branch (*B1*, *B2*, *B3*, and *B4*) shows that the left-hand side of equation (F.2) above is always less than or equal to the right-hand side:

$$\begin{aligned} &\underline{\mathbf{B1}} \quad \left(q^{\omega}-p_a^{\omega}\right)g \ge 0, \textit{ and } \left(q^{\omega}-p_b^{\omega}\right)g \ge 0 \\ &\quad \Rightarrow f_{\omega 1}\left(\lambda p_a^{\omega}+(1-\lambda)p_b^{\omega}\right) = \lambda f_{\omega 1}\left(p_a^{\omega}\right)+(1-\lambda)f_{\omega 1}\left(p_b^{\omega}\right) \end{aligned} \qquad \textbf{(F.5)}$$

$$\begin{aligned} &\underline{\mathbf{B2}} \quad \left(q^{\omega}-p_a^{\omega}\right)g \ge 0 \ge \left(q^{\omega}-p_b^{\omega}\right)g \\ &\quad \Rightarrow \lambda\left(q^{\omega}-p_a^{\omega}\right)g+(1-\lambda)\left(q^{\omega}-p_b^{\omega}\right)g \le \lambda\left(q^{\omega}-p_a^{\omega}\right)g \\ &\quad \Rightarrow f_{\omega 1}\left(\lambda p_a^{\omega}+(1-\lambda)p_b^{\omega}\right) \le \lambda f_{\omega 1}\left(p_a^{\omega}\right)+(1-\lambda)f_{\omega 1}\left(p_b^{\omega}\right) \end{aligned} \qquad \textbf{(F.6)}$$

$$\begin{aligned} &\underline{\mathbf{B2}} \quad \left(q^{\omega}-p_b^{\omega}\right)g \ge 0 \ge \left(q^{\omega}-p_a^{\omega}\right)g \\ &\quad \Rightarrow \lambda\left(q^{\omega}-p_b^{\omega}\right)g+(1-\lambda)\left(q^{\omega}-p_a^{\omega}\right)g \le \lambda\left(q^{\omega}-p_b^{\omega}\right)g \\ &\quad \Rightarrow f_{\omega 1}\left(\lambda p_b^{\omega}+(1-\lambda)p_a^{\omega}\right) \le \lambda f_{\omega 1}\left(p_b^{\omega}\right)+(1-\lambda)f_{\omega 1}\left(p_a^{\omega}\right) \end{aligned} \qquad \textbf{(F.7)}$$

$$\begin{aligned} &\underline{\mathbf{B4}} \quad \left(q^{\omega}-p_a^{\omega}\right)g \le 0, \textit{ and } \left(q^{\omega}-p_b^{\omega}\right)g \le 0 \\ &\quad \Rightarrow f_{\omega 1}\left(\lambda p_a^{\omega}+(1-\lambda)p_b^{\omega}\right) = \lambda f_{\omega 1}\left(p_b^{\omega}\right)+(1-\lambda)f_{\omega 1}\left(p_a^{\omega}\right) = 0 \end{aligned} \qquad \textbf{(F.8)}$$

Therefore, $f_{\omega 1}$ is convex. By the same manner of proof $f_{\omega 3}$ and $f_{\omega 4}$ can also be shown to be convex. Further, $f_{\omega 2}$ is convex since any linear function is convex. Therefore, as each $f_{\omega i}$ is convex, *L(p)* is convex.

Q.E.D.

Proof of Lemma 4.1:

The first step in determining the necessary and sufficient conditions for the restricted (date 12 payment only) problem is to demonstrate that the problem can be decomposed into sub-problems which can be solved independently. From equation (4.4), the problem may be written as one of minimizing the sum of miniature objective functions, one for each date 12 event,

$$\begin{aligned} \min_{\tilde{p}} L(\tilde{p}) &= \min_{\tilde{p}} \sum_{j=1}^{J_{12}} \sum_{\omega \in \Phi_{12j}} Prob_{\omega} A_j(\tilde{p}_{12j}) \\ &= \sum_{j=1}^{J_{12}} \min_{\tilde{p}} \sum_{\omega \in \Phi_{12j}} Prob_{\omega} A_j(\tilde{p}_{12j}) \end{aligned} \tag{F.9}$$

where,

$$\begin{aligned} A_j(\tilde{p}) = \sum_{\omega \in \Omega} \Bigg\{ & \max\left[0, \left(\sum_{i=1}^{12} q_i^{\omega} g_i\right) - p_{12j} g_i\right] + p_{12j} c_{12} - \sum_{i=1}^{12} q_i^{\omega} c_i \\ & + .50 \cdot \max\left[0, \left(\sum_{i=1}^{12} q_i^{\omega} g_i\right) - p_{12j} g_{12} - \max\left(1000, .25 \sum_{i=1}^{12} q_i^{\omega} g_i\right)\right] \\ & + \max\left[0, p_{12j} s_{yz} - \left(\sum_{i=1}^{12} x^{\omega} s_{yz}\right)\right] \Bigg\} Prob_{\omega} \end{aligned} \tag{F.10}$$

As each of these miniature objective functions $A_j(\tilde{p}_j)$ is a function of $\tilde{p}_j$ alone (*i.e.*, it is not a function of any contingent payment $\tilde{p}_{k \neq j}$), the individual problems in equation (F.9) may be solved independently. For example, the *j*th problem to be solved is,

$$\min_{\bar{p}_{12j}} \sum_{\omega \in \phi_{12j}} Prob_{\omega} A_j(\bar{p}_{12j}) \tag{F.11}$$

As discussed in section 4.3 of chapter 4, a necessary and sufficient condition for this problem is that the one-sided directional derivative of the objective function set out in equation (F.11), be non-negative in all feasible directions. Consider first the interior solution $\bar{p}_{12j} > 0$ as an optimum. Two obvious feasible directions are $v\uparrow$ and $v\downarrow$, where,

$$v\uparrow = (0, 0, \ldots, 0, 1, 0, \ldots 0) \tag{F.12}$$

and

$$v\downarrow = (0, 0, \ldots, 0, -1, 0, \ldots 0) \tag{F.13}$$

where the non-zero element corresponds to the *j*th event in December.

The direction $v\uparrow$ is examined first. The corresponding one-sided directional derivative is, from equation (4.28),

$$\sum_{\omega \in \phi_{12j}} Prob_{\omega} \cdot (1) \cdot \left(I_{i\omega} + c_i + Pen_{i\omega}\right) \tag{F.14}$$

Since $I_{i\omega}$ and $Pen_{i\omega}$ can take different values, it is convenient to group the states by their relationship to $I_{i\omega}$ and $Pen_{i\omega}$. For the five groups, Ω_1 to Ω_5, defined in equations (4.32) to

(4.36), the derivative above simplifies to,

$$L'(\bar{p};v\uparrow) = prob_1(-1.5g_{12}+c_{12}) + (prob_2+prob_3)(-g_{12}+c_{12}) + (prob_4+prob_5)\,c_{12} \tag{F.15}$$

where $prob_l,\ l=1,2,3,4,5$ is defined in equation (4.37). The Diewert condition that this expression be non-negative is the right-hand side of equation (4.38) of the text. The left-hand side of equation (4.38) is derived similarly by considering the feasible direction $v\downarrow$ defined in equation (F.13) above. The associated direction is:

$$L'(\bar{p};v\downarrow) = (Prob_1+Prob_2)(1.5g_{12}-c_{12}) + (Prob_3+Prob_4)(g_{12}-c_{12}) + Prob_5(-c_{12}) \tag{F.16}$$

Where $\bar{p}_{12j} = 0$, the obvious feasible direction is,

$$v = (0,0,\ldots,0,1,0,\ldots 0) \tag{F.17}$$

and the directional derivative equals equation (F.15). The necessary condition for an optimum is that this derivative be non-negative, which is the condition set out in equation (4.38).

The Diewert condition requires that all feasible directions be examined, not just $v\uparrow$ and $v\downarrow$. However, all other feasible directions are scalar multipliers of equations (F.15) and (F.16). Therefore, applying the Diewert condition that these be non-negative does not yield any new information. Hence, it is sufficient to examine only the Diewert conditions for $v\uparrow$ and $v\downarrow$.

Q.E.D.

Proof of Lemma 4.2:

Let us prove that any payment structure may be replicated in terms of its objective value by a payment structure with only December payments. That is, for any payment vector p, there exists a payment vector $\tilde{p}$ with all non-December payments zero and with the same expected loss.

The expected loss function for $L(p)$, restricting the loss function defined in equation (4.3) above such that there is no stub loss, may be written,

$$L(p) = \sum_{\omega\in\Omega} \left\{ \max\left[0, \sum_{i=1}^{12} (q_i^\omega - p_i^\omega) g_i\right] + \sum_{i=1}^{12} (p_i^\omega - q_i^\omega) c_i \right.$$
$$\left. + .50 \cdot \max\left[0, \sum_{i=1}^{12} (q_i^\omega - p_i^\omega) g_i - \max\left(1000, .25 \sum_{i=1}^{12} q_i^\omega g_i\right)\right]\right\} Prob_\omega \qquad \text{(F.18)}$$

The expected loss for the replicating payment path $\tilde{p}$ may similarly be written,

$$L(\tilde{p}) = \sum_{\omega\in\Omega} \left\{ \max\left[0, \sum_{i=1}^{12} q_i^\omega g_i - p_i^\omega g_{12}\right] + \left(p_{12}^\omega c_{12} - \sum_{i=1}^{12} q_i^\omega c_i\right) \right.$$
$$\left. + .50 \cdot \max\left[0, \left(\sum_{i=1}^{12} q_i^\omega g_i - p_{12}^\omega g_{12}\right) - \max\left(1000, .25 \sum_{i=1}^{12} q_i^\omega g_i\right)\right]\right\} Prob_\omega \qquad \text{(F.19)}$$

Substituting the definition of $\tilde{p}$ in equation (4.46) into equation (F.19), and then subtracting the resulting amount from equation (F.18) yields,

$$L(p) - L(\bar{p})$$

$$= \sum_{\omega\in\Omega}\left(\sum_{i=1}^{12} p_i^{\omega} c_i - \sum_{i=1}^{12} q_i^{\omega} c_i\right) Prob_{\omega} - \sum_{\omega=1}^{\Omega}\left(\frac{\sum_{i=1}^{12} p_i^{\omega} g_i}{g_{12}} \cdot c_{12} - \sum_{i=1}^{12} q_i^{\omega} c_i\right) Prob_{\omega} \tag{F.20}$$

as terms other than the second terms in equations (F.18) and (F.19) (the opportunity loss terms) cancel directly. With the interest rate assumptions,[74]

$$\frac{\sum_{i=1}^{12} p_i^{\omega} g_i}{g_{12}} \cdot c_{12} = \sum_{i=1}^{12} p_i^{\omega} c_i \tag{F.21}$$

[74]**Proof:** With simple and constant rates,

$$g_i = \frac{\sum_{k=i+1}^{13} N_k}{365} \cdot G \quad and \quad c_i = \frac{\sum_{k=i+1}^{13} N_k}{365} \cdot C$$

Hence:

$$\frac{\sum_{i=1}^{12} p_i^{\omega} g_i}{g_{12}} c_{12} = \frac{\sum_{i=1}^{12} p_i^{\omega} \cdot \frac{\sum_{k=i+1}^{13} N_k}{365} G}{\frac{N_{13}}{365} G} \cdot \frac{N_{13}}{365} C$$

$$= \sum_{i=1}^{12} p_i^{\omega} \frac{\sum_{k=i+1}^{13} N_k}{365} \cdot C$$

$$= \sum_{i=1}^{12} p_i^{\omega} c_i$$

and therefore,

$$L(p) - L(\tilde{p}) = 0. \tag{F.22}$$

Q.E.D.

Proof of Proposition 4.1:

The proof has two parts: necessity and sufficiency.

(a) Sufficiency:

Sufficiency will be proven in two steps: first it will be demonstrated that $\tilde{p}^*$ is an optimum for the unrestricted problem (minimizing equation (4.3)); and second, that any vector p which satisfies the right-hand side of equation (4.48) is an optimum for the unrestricted problem.

Step 1: Proof that $\tilde{p}^*$ is an optimum for the unrestricted problem

Suppose that $\tilde{p}^*$ is not an optimum for the unrestricted problem. Then there must exist some payment vector $\bar{p}$ with a lower objective function value. However, if such a vector $\bar{p}$ exists, then by replication (Lemma 4.2) there exists a payment vector $\tilde{\bar{p}}$ (with no non-date 12 payments) which has the same lower objective function value. Therefore, $\tilde{p}^*$ cannot be an optimum for the problem of minimizing equation (4.3) subject to the restriction that all non-date 12 payments equal zero (the restricted problem), which contradicts Lemma 4.1. Therefore, the assumption that $\tilde{p}^*$ is not an optimum for the unrestricted problem must be wrong.

<u>Step 2: Any vector p which satisfies the right-hand side of equation (4.48) is an optimum for the unrestricted problem</u>

By step 1 of this proof, $\bar{p}^*$ is an optimum for the main problem. Further, as equation (4.46) implies the right-hand side of equation (4.48), by the replication result (Lemma 4.2 above) any point which satisfies equation (4.48) is also an optimum for the unrestricted problem.

Q.E.D.

(b) Necessity:

Suppose that there exists a payment vector $\bar{\bar{p}}$ which does not satisfy equation (4.48) but is an optimum for the unrestricted problem. By Lemma 4.2, there exists a vector $\bar{\bar{p}} \neq \bar{p}^*$ which is an optimum for the restricted problem. Therefore, equations (4.38) and (4.39) are not necessary conditions for the restricted problem. Therefore, Lemma 4.1 is wrong, which is a contradiction.

Q.E.D.

Proof of Corollary 4.1:

The proof of Corollary 4.1 takes two steps. The first step is to prove that ω is a member of Ω_4. The second step is to demonstrate that equation (4.54) in chapter 4 satisfies Proposition 4.1.

Step 1:

The first step is to characterize $\bar{p}^*$. Note that ω must belong to one of Ω_1 through Ω_5 set out in equations (4.32) through (4.36); it cannot belong to more than one as these are mutually exclusive.

Let us demonstrate this proposition through contradiction. Assume,

$$\omega \in \Omega_1 \rightarrow Prob_1 = 1, \; Prob_2 \; to \; Prob_5 = 0 \tag{F.23}$$

Substituting these probabilities into equations (4.38) and (4.39), the necessary and sufficient conditions for $\bar{p}$ to be an optimum reduce to,

$$\begin{cases} 0 \leq -1.5g_{12} + c_{12} & \text{if } \bar{p}_j = 0 \\ -1.5g_{12} + c_{12} \leq 0 \leq -1.5g_{12} + c_{12} & \text{if } \bar{p}_j > 0 \end{cases} \tag{F.24}$$

which is false as, by assumption, $g_{12} > c_{12}$ and therefore $1.5g_{12} > c_{12}$. Similar results follow for Ω_2, Ω_3, and Ω_5.

Now assume $\omega \in \Omega_4$, which implies that $I_\omega = 0$. Since

$$\omega \in \Omega_4 \rightarrow Prob_4 = 1, \; Prob_1, Prob_2, Prob_3, Prob_5 = 0 \tag{F.25}$$

The necessary and sufficient conditions for $\bar{p}$ to be an optimum are, from equations (4.38) and (4.39),

$$\begin{cases} 0 \leq c_{12} & \text{if } \bar{p}_j = 0 \\ -g_{12} + c_{12} \leq 0 \leq c_{12} & \text{if } \bar{p}_j > 0 \end{cases} \qquad \text{(F.26)}$$

These conditions are satisfied since by assumption $g_{12} > c_{12}$ and $c_{12} > 0$.

<u>Step 2:</u>

As,

$$\Omega_4 = \{ \omega \mid Pen_\omega < 0, I_\omega = 0 \} \qquad \text{(F.27)}$$

and $\bar{p}$ has all non-date 12 payments equal to zero, $\bar{p}^*$ is defined by the equations,

$$\sum_{i=1}^{12} p_i g_i = \sum_{i=1}^{12} q_i g_i \quad \text{and} \quad p_{ij} = 0 \ \ \forall\, i < 12 \qquad \text{(F.28)}$$

where the first condition is defined by $I_\omega = 0$ and the second condition by the requirement of no payments prior to date 12. Note that to simplify notation, the superscript ω has been omitted, as there is only a single state. These conditions are jointly satisfied where,

$$\bar{p}^* = \left[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, \frac{\sum_{i=1}^{12} q_i g_i}{g_{12}} \right] \qquad \text{(F.29)}$$

Hence, applying Proposition (4.1) above and noting that any point which satisfies the left-hand side of equation (4.48) also satisfies the right hand side of equation (4.48), the corollary is proved.

Lemma 4.3:

$$\text{If } p_j = 0 \ \ (j \geq 2), \ \text{ then } I_\omega \geq 0 \ \Rightarrow S_\omega < 0$$

Proof:

$$
\begin{aligned}
& I_\omega \geq 0 \\
\Rightarrow\ & \sum_{i=1}^{12} q_i g_i - \sum_{i=1}^{12} p_i g_i \ \geq 0 \\
\Rightarrow\ & \sum_{i=1}^{12} q_i g_i - p_1 g_1 \ \geq 0 \quad \text{since } p_j = 0,\ j \geq 2 \\
\Rightarrow\ & \sum_{i=1}^{12} q_i \frac{g_i}{g_1} - p_1 \ \geq 0 \quad \text{since } g_1 > 0 \\
\Rightarrow\ & \sum_{i=1}^{12} q_i - p_i \ \geq 0 \quad \text{since } g_1 > g_i \ \ \forall\, i \neq 1 \\
\Rightarrow\ & \sum_{i=1}^{12} q_i - \sum_{i=1}^{12} p_i \ \geq 0 \quad \text{since } p_j = 0, j \geq 2 \\
\Rightarrow\ & S_\omega \ < 0
\end{aligned}
$$

Proof of Necessity for Proposition 4.3:

Where the rates g_i and c_i are simple, non-stochastic, and unchanging any time path of payments which does not satisfy equations (4.55) and (4.56) is not optimal. As stated in chapter 4, this is proven through demonstrating:

(*a*) where $I_\omega < 0$, there exists a direction v in which $L'(p_0; v) < 0$;

(*b*) where $I_\omega > 0$, there exists a direction v in which $L'(p_0; v) < 0$;

(*c*) where $I_\omega = 0$ and $S_\omega > 0$, there exists a direction v in which $L'(p_0; v) < 0$;

(*a*) Suppose $I_\omega < 0$: that is, there is overpayment with respect to instalment interest

$I_\omega < 0$ implies that there exists a $p_j > 0$ for some $j = 1, 2, \ldots, 12$. Hence, the direction vector,

$$v = (0, \ldots, 0, -1, 0, \ldots 0)$$

where the non-zero element which is the *j*th element, is a feasible direction. The one-sided directional derivative in this direction is,

$$\begin{aligned} L'(p_0; v) &= \sum_{i=1}^{12} v_i \left(I_{i\omega} + c_i + Pen_{i\omega} + S_{i\omega}\right) \\ &= I_{i\omega} + c_i + Pen_{i\omega} + S_{i\omega} \\ &= 0 + c_i + 0 + S_{i\omega} \end{aligned} \tag{F.24}$$

as, from equation (4.22) and (4.25), $I_{i\omega}$ and $Pen_{i\omega} = 0$. Therefore,

where these two branches cover all possibilities as $I_\omega < 0 \rightarrow Pen_\omega < 0$. Therefore, as $c_k > 0$

$$L'(p_0; v_{pi1}) = \begin{cases} -c_k - s_{yz} & \text{if } Pen_\omega < 0,\ I_\omega < 0,\ \text{and } S_\omega > 0 \\ -c_k & \text{if } Pen_\omega < 0,\ I_\omega < 0,\ \text{and } S_\omega \le 0 \end{cases} \quad \text{(F.25)}$$

and $s_{yz} \ge 0$,

$$L'(p_0; v_{pi1}) < 0 \quad \text{(F.26)}$$

(*b*) Suppose $I_\omega > 0$: that is, there is underpayment with respect to instalment interest

Two cases may be examined: first, where the corporation makes a positive payment at any payment date after January; and second, where the corporation only makes a positive payment in January (at its first instalment date). It will be demonstrated that in the first case, the corporation can reduce its loss by shifting amounts to the first payment date. In the second case, the corporation can reduce its loss through increasing its January payment.

Case 1: Let us decrease the payment at payment date $j > 1$ by \$1 and increase the payment at payment date 1 by \$1: the effect is to reduce underpayment with respect to instalment interest while holding the stub loss constant.

Consider the direction v defined by,

$$\begin{aligned} &v_1 = 1,\ v_j = -1 \\ &v_i = 0 \quad \forall\ i \notin \{1, j\} \end{aligned} \quad \text{(F.27)}$$

The one-sided directional derivative, from equation (4.28), is,

$$L'(p_0; v) = \sum_{i=1}^{12} v_i \left(I_{i\omega} + c_i + Pen_{i\omega} + S_{i\omega}\right) \tag{F.28}$$

which may be rewritten,

$$\begin{aligned} L'(p_0; v) &= I_{1\omega} + c_1 + Pen_{1\omega} + S_{1\omega} \quad - \left(I_{j\omega} + c_j + Pen_{j\omega} + S_{j\omega}\right) \\ &= \begin{cases} -1.5\, g_1 + c_1 \; + \; 1.5\, g_j - c_j & \text{if} \quad Pen_\omega > 0 \\ -g_1 + c_1 \; + \; g_j - c_j & \text{if} \quad Pen_\omega \leq 0 \end{cases} \end{aligned} \tag{F.29}$$

as $I_\omega > 0$. Note that as $S_{k\omega} - S_{j\omega} = s_{yz} - s_{yz} = 0$, the stub amount did not enter the expression. The amount in equation (F.29) is negative as, by assumption, $g_i > c_i$. Hence, there exists a direction v in which $L'(p_0; v) < 0$.

<u>Case 2:</u> Let us increase the payment at date 1 by \$1: the effect is to reduce underpayment with respect to instalment interest while holding the stub loss constant.

Consider the direction v defined by,

$$v_1 = 1 \tag{F.30}$$

The one-sided directional derivative, from equation (4.28), is,

$$
\begin{aligned}
L'(p_0; v) &= I_{1\omega} + c_1 + Pen_{1\omega} + S_{1\omega} \\
&= \begin{cases} -1.5\, g_1 + c_1 & \text{if} \quad Pen_{1\omega} > 0 \\ -g_1 + c_1 & \text{if} \quad Pen_{1\omega} \le 0 \end{cases}
\end{aligned}
\tag{F.31}
$$

as $I_\omega > 0$ and as, by Lemma 4.3, $S_\omega < 0$. The amount in equation (F.31) is negative as, by assumption, $g_i > c_i$.

(c) Suppose $I_\omega = 0$ and $S_\omega > 0$

For the conditions $I_\omega = 0$ and $S_\omega > 0$ to hold jointly, a positive payment amount must exist at some date $k > 1$ and therefore the following direction is feasible,

$$
\begin{aligned}
&v_k = -1, \quad v_1 = \frac{g_k}{g_1} \quad \textit{where} \quad k > 1 \\
&v_i = 0 \quad \forall \ i \notin \{k, 1\}
\end{aligned}
\tag{F.32}
$$

In other words, let us decrease the payment at payment date k by \$1 and increase the payment at payment date 1 by $\frac{g_k}{g_1}$. The effect is that underpayments with respect to instalment interest are held constant while the stub loss is decreased.

The one-sided directional derivative, from equation (4.28), is,

$$
L'(p_0; v) = \sum_{i=1}^{12} v_i \left(I_{i\omega} + c_i + Pen_{i\omega} + S_{i\omega} \right) \tag{F.33}
$$

which may be rewritten,

$$L'(p_0; v) = -(I_{k\omega} + c_k + Pen_{k\omega} + S_{k\omega}) + \frac{g_k}{g_1}(I_{1\omega} + c_1 + Pen_{1\omega} + S_{1\omega})$$

$$= -c_k - S_{yz} + \frac{g_k}{g_1}(c_1 + S_{yz}) \qquad \textbf{(F.34)}$$

$$= -S_{yz} + \frac{g_k}{g_1} S_{yz}$$

as $c_i = \frac{g_k}{g_1} c_1$ where rates are simple. This amount is less than zero as $0 < \frac{g_k}{g_1} < 1$ (as $g_t <$ g_s for all $t > s$). Therefore, where $I_\omega = 0$ and $S_\omega > 0$, there exists a direction v in which $L'(p_0; v) < 0$.

Q.E.D.

Lemma 4.4:

$$c_1 > \frac{g_1}{g_j} c_1 \quad or \quad \frac{g_j}{g_1} > \frac{c_j}{c_1}$$

Restating the Problem:

We want to demonstrate that,

$$\frac{(a^d - 1)}{(a^b - 1)} > \frac{(c^d - 1)}{(c^b - 1)}$$

$$where \quad a = \left(1 + \frac{G}{365}\right) > 1$$

$$b = \sum_{i=j+1}^{13} N_i > 1$$

$$c = \left(1 + \frac{C}{365}\right) > 1$$

$$d = \sum_{i=2}^{13} N_i > 1$$

where $\sum_{i=2}^{13} N_i$ is the number of days from the first payment date to the remainder due date and $\sum_{i=j+1}^{13} N_i$ is the number of days from the *j*th payment date to the remainder due date. Note that by construction $a > c$ (as G is assumed to be greater than C) and $d > b$ (the number of days from payment date $j \geq 2$ to the remainder due date is less than the number of days from the first payment date to the remainder due date).

Note that since $a > c$, the equation directly above is equivalent to stating that $f(y)$ is strictly increasing for $y > 1$ where,

$$f(y) = \frac{y^d - 1}{y^b - 1}$$

That is, this lemma is proven if,

$$f'(y) > 0$$

Let us now examine the derivative of the function,

$$\begin{aligned} f'(y) &= \frac{(y^b - 1)(dy^{d-1}) - (y^d - 1)(by^{b-1})}{(y^b - 1)^2} \\ &= \frac{y^{b+d-1}(d-b) - dy^{d-1} + by^{b-1}}{(y^b - 1)^2} \\ &= \left[\frac{y^{b-1}}{(y^b - 1)^2}\right]\left((d-b)y^d - dy^{d-b} + b\right) \\ &= \left[\frac{y^{b-1}}{(y^b - 1)^2}\right] g(y) \qquad \textit{where} \quad g(y) = (d-b)y^d - dy^{d-b} + b \end{aligned}$$

Therefore,

$$f'(y) > 0 \quad \textit{if} \quad g(y) > 0$$

To demonstrate that $g(y) > 0$ for $y > 1$, let us demonstate that:

a. $g(1) = 0$;

b. $g(\infty) = \infty$; and

c. there are no turning points for values of $y > 1$.

a. $g(1) = 0$

$$g(1) = d - b - d + b = 0$$

b. $g(\infty) = \infty$

Let us define a relationship $d = b + \epsilon$ where $\epsilon > 0$ and $b > 1$.

$$\begin{aligned} \lim_{y\to\infty} g(y) &= \lim_{y\to\infty} \left(\epsilon y^{b+\epsilon} - (b+\epsilon) y^{\epsilon} + b\right) \\ &= \infty \end{aligned} \quad \text{(F.42)}$$

c. there are no turning points for values of $y > 1$

The turning points of $g(y)$ occur where $g'(y) = 0$, where

$$\begin{aligned} g'(y) &= \epsilon(b+\epsilon) y^{b+\epsilon-1} - \epsilon(b+\epsilon) y^{\epsilon-1} \\ &= \epsilon(b+\epsilon) y^{\epsilon-1}\left[y^{b} - 1\right] \end{aligned} \quad \text{(F.43)}$$

The zeros are, $y = 0$ and $y^{b} = 1 \rightarrow y = 1$. As none of the zeros are in $(1, \infty)$, $g(y)$ has no turning points for $y > 1$.

Q.E.D.

Lemma 4.5:

Given $P_\omega \geq 0$, *then* $S_\omega < 0$ for reasonable values of G (that is, for government interest rates less than about 30%).

Proof:

Consider the problem,

$$\max_p S_\omega \quad s.t. \quad P_\omega = \overline{P_\omega} \quad or$$

$$.5\left[\sum_{i=1}^{12} q_i g_i - \sum_{i=1}^{12} p_i g_i - \max\left(1000, .25\sum_{i=1}^{12} q_i g_i\right)\right] = \overline{P_\omega}$$

where $\overline{P_\omega}$ is a fixed amount. The solution to this problem is,

$$\sum_{i=1}^{12} q_i g_i - p_{12} g_{12} - \max(\cdot) = \frac{\overline{P_\omega}}{.5}$$

$$\sum_{i=1}^{12} q_i g_i - \max(\cdot) - \frac{\overline{P_\omega}}{.5} = p_{12} g_{12}$$

$$p_{12} = \frac{\sum_{i=1}^{12} q_i g_i - \max(\cdot) - \frac{\overline{P_\omega}}{.5}}{g_{12}}$$

Therefore,

$$
\begin{aligned}
& P_\omega = \overline{P_\omega} \\
& \Rightarrow S_\omega \leq \textit{its maximum possible value} \\
& \Rightarrow S_\omega \leq s_{yz}\left(\sum_{i=1}^{12} p - \sum_{i=1}^{12} x\right) \\
& \Rightarrow S_\omega \leq \frac{s_{yz}}{g_{12}}\left(\sum_{i=1}^{12} q_i g_i - \max(\cdot) - \frac{\overline{p_\omega}}{.5} - \sum_{i=1}^{12} x g_{12}\right) \\
& \Rightarrow S_\omega \leq \frac{s_{yz}}{g_{12}}\left(.75\sum_{i=1}^{12} q_i g_i - \frac{\overline{p_\omega}}{.5} - \sum_{i=1}^{12} x g_{12}\right) \\
& \Rightarrow S_\omega \leq \frac{s_{yz}}{g_{12}}\left(.75\sum_{i=1}^{12} g_i x - \frac{\overline{p_\omega}}{.5} - \sum_{i=1}^{12} x g_{12}\right) \quad \textit{since} \quad q_i \leq x \\
& \Rightarrow S_\omega \leq \frac{s_{yz}}{g_{12}}\left(.75\sum_{i=1}^{12} g_i x - \sum_{i=1}^{12} x g_{12}\right) < 0
\end{aligned}
$$

for interest rates less than about 30%. This follows as $x > 0$. That x must be positive may be demonstrated as follows: Suppose,

$$
\begin{aligned}
& x = 0 \\
& \Rightarrow q_i = 0 \ \forall \ i \\
& \Rightarrow P_\omega < 0
\end{aligned}
$$

which contradicts our assumption.

Lemma 4.6

Given $S_\omega > 0$, *then* $P_\omega < 0$ for reasonable values of G (that is, for government interest rates less than about 30%).

Proof:

Consider the problem,

$$\max_{p} P_\omega \ \ s.t. \ S_\omega = \overline{S_\omega} \ \ where \ \overline{S_\omega} > 0$$

where $\overline{S_\omega}$ is a fixed amount. Rewriting the constraint,

$$s_{yz}\sum_{i=1}^{12}(p_i - x) = \overline{S_\omega}$$
$$\Rightarrow \sum_{i=1}^{12}(p_i - x) = \frac{\overline{S_\omega}}{s_{yz}}$$
$$\Rightarrow \sum_{i=1}^{12} p_i = \frac{\overline{S_\omega}}{s_{yz}} + \sum_{i=1}^{12} x$$

From the Kuhn-Tucker conditions, the solution is,

$$p_{12} = \frac{\overline{S_\omega}}{s_{yz}} + \sum_{i=1}^{12} x > 0$$
$$p_i = 0 \ \ \forall \, i \neq 0$$

Therefore,

$$S_\omega = \overline{S_\omega}$$

$$\rightarrow P_\omega \le maximum$$

$$\rightarrow P_\omega \le \sum_{i=1}^{12} q_i g_i - g_{12}\left[\frac{\overline{S_\omega}}{s_{yz}} + \sum_{i=1}^{12} x - \max(\cdot)\right]$$

$$\rightarrow P_\omega \le .75\sum_{i=1}^{12} q_i g_i - g_{12}\left[\frac{\overline{S_\omega}}{s_{yz}} + \sum_{i=1}^{12} x\right]$$

$$\rightarrow P_\omega \le .75\sum_{i=1}^{12} g_i x - g_{12}\left[\frac{\overline{S_\omega}}{s_{yz}} + \sum_{i=1}^{12} x\right]$$

$$\rightarrow P_\omega \le .75\sum_{i=1}^{12} g_i x_i - g_{12}\sum_{i=1}^{12} x$$

$$\rightarrow P_\omega \le x\left(.75\sum_{i=1}^{12} g_i - 12g_{12}\right)$$

$$\rightarrow P_\omega \le x\sum_{i=1}^{12}\left(.75g_i - g_{12}\right) < 0 \qquad \textit{if } g_i < \frac{4}{3}g_{12}$$

which is true for interest rates less than about 30%.

Lemma 4.7:

$$P_\omega > 0 \rightarrow I_\omega > 0$$

Proof:

$$P_\omega = .5\, I_\omega - \max\left(1000,\ .25 \sum_{i=1}^{12} q_{i\omega}\, g_i\right)$$

$$\therefore\ P_\omega > .5\, I_\omega$$

$$\therefore\ P_\omega > 0 \quad \rightarrow\ I_\omega > 0$$

Proof of Proposition 4.4:

As the expected loss function set out in equations (4.6) to (4.13) reduces to a function with a single state with probability one under certainty, the endogenous variable are $p = (p_1, p_2, ..., p_{12})$ and the directions are of the form $v = (v_1, v_2, ..., v_{12})$. Consider the direction v defined by,

$$\begin{aligned} v_1 &= 1 \\ v_j &= -\frac{g_1}{g_j} \quad where \;\; 2 \le j \le 12 \\ v_i &= 0 \quad \forall \;\; i \notin \{1,j\} \end{aligned} \tag{F.35}$$

The one-sided directional derivative, from equation (4.28), is,

$$\begin{aligned} L'(p_0; v) &= \sum_{\omega=1}^{\Omega} Prob_\omega \sum_{i=1}^{12} v_i \left(I_{i\omega} + c_i + P_{i\omega} + S_{i\omega}\right) \\ &= \sum_{i=1}^{12} v_i \left(I_{i\omega} + c_i + P_{i\omega} + S_{i\omega}\right) \end{aligned} \tag{F.36}$$

This occurs as a single state of nature occurs with probability one. Equation (F.36) may be rewritten,

$$\begin{aligned} L'(p_0; v) &= I_{1\omega} + c_1 + P_{1\omega} + S_{1\omega} - \frac{g_1}{g_j}\left(I_{j\omega} + c_j + P_{j\omega} + S_{j\omega}\right) \\ &= c_1 - \frac{g_1}{g_j} c_j + S_{1\omega} - \frac{g_1}{g_j} S_{j\omega} \end{aligned} \tag{F.37}$$

since from equations (4.22) and (4.25) $I_{i\omega} - \frac{g_1}{g_j} I_{j\omega} = P_{i\omega} - \frac{g_1}{g_j} P_{j\omega} = 0$, *i.e.*, in this direction v the penalty, P_ω, and instalment interest, I_ω, are held constant. Equation (F.37) may be rewritten,

$$L'(p_0; v) = \begin{cases} c_1 - \frac{g_1}{g_j} c_j + \left(1 - \frac{g_1}{g_j}\right) S_{y\pi} & \text{if } S_\omega > 0 \\ c_1 - \frac{g_1}{g_j} c_j & \text{if } S_\omega \le 0 \end{cases} \tag{F.38}$$

This amount is less than zero where $c_1 > \frac{g_1}{g_j} c_j$ (see Lemma 4.4) and $g_1 > g_j$. Therefore, increasing p_1 by \$1 and decreasing p_j by $\frac{g_1}{g_j}$ decreases the objective function. Since this is a feasible direction whenever there exists a $p_j > 0$, it follows that in the optimum $p_j = 0 \;\; \forall j = 2, 3, \ldots, 12$. In other words, the only instalment payment which can be non-negative is the one paid in the first month of the year.

Now consider the direction $v_{p1\uparrow}$ in which p_1 increases and all other payments stay constant:

$$\begin{aligned} v_1 &= 1 \\ v_j &= 0, \quad 2 \le j \le 12 \end{aligned} \tag{F.39}$$

The one-sided directional derivative in this direction is, from (4.28)

$$L'(p_0; v_{p1\uparrow}) = \sum_{i=1}^{12} v_i \left(I_{i\omega} + c_i + P_{i\omega} + S_{i\omega}\right)$$

$$= I_{1\omega} + c_1 + P_{1\omega} + S_{1\omega}$$

$$= \begin{cases} -1.5\,g_1 + c_1 & \text{if } P_\omega > 0,\ I_\omega > 0,\ \text{and } S_\omega < 0 \\ -g_1 + c_1 & \text{if } P_\omega \le 0,\ I_\omega > 0,\ \text{and } S_\omega < 0 \\ c_1 + S_{yz} & \text{if } P_\omega < 0,\ I_\omega \le 0,\ \text{and } S_\omega \ge 0 \\ c_1 & \text{if } P_\omega < 0,\ I_\omega \le 0,\ \text{and } S_\omega < 0 \end{cases} \tag{F.40}$$

Note that from Lemmas 4.3, 4.5, 4.6, and 4.7, the four branches of the definition above cover all values of P_ω, I_ω, and S_ω which can occur. Similarly, consider the direction $v_{p1\downarrow}$ in which p_1 decreases and all other payments stay constant:

$$\begin{aligned} v_1 &= -1 \\ v_j &= 0, \quad 2 \le j \le 12 \end{aligned} \tag{F.41}$$

Following the same argument as above, the one-sided directional derivative in this direction is,

$$L'(p_0; v_{p1\downarrow}) = \begin{cases} 1.5\,g_1 - c_1 & \text{if } P_\omega > 0,\ I_\omega > 0,\ \text{and } S_\omega < 0 \\ g_1 - c_1 & \text{if } p_\omega \le 0,\ I_\omega > 0,\ \text{and } S_\omega < 0 \\ -c_1 - S_{yz} & \text{if } P_\omega < 0,\ I_\omega \le 0,\ \text{and } S_\omega \ge 0 \\ -c_1 & \text{if } P_\omega < 0,\ I_\omega \le 0,\ \text{and } S_\omega < 0 \end{cases} \tag{F.42}$$

where the four branches again cover all possibilities.

Using these derivatives, it can be shown that the payment structure,

$$p_1 = \frac{\sum_{i=1}^{12} q_i g_i}{g_1} \qquad \text{(F.43)}$$

$$p_j = 0, \quad j \geq 2$$

satisfies the Diewert conditions for an optimum. Since it is shown above that $p_j = 0$ (for all $j \geq 2$), there are only two feasible directions: $v_{p1\uparrow}$ and $v_{p1\downarrow}$.

For this payment structure $I_\omega = 0$ and, from Lemma 4.3, $S_\omega < 0$, so,

$$v_{p1\uparrow} = c_1 \quad > \quad 0 \qquad \text{(F.44)}$$

Similarly,

$$v_{p1\downarrow} = g_1 - c_1 \quad > \quad 0 \qquad \text{(F.45)}$$

Q.E.D.

APPENDIX G

NOTATION

		First Used at Page
AETR	- the corporation's average effective tax rate under the instalment structure	136
B_1	- the corporation's tax liability for the first preceding year	39
b_1	- 1/12 of the corporation's tax liability for the first preceding year	20
b_2	- 1/12 of the corporation's tax liability for the second preceding year	20
C_i	- the corporation's after tax cost of capital for period *i*. Where rates are constant, and in the single instalment model in chapter 3, the subscript is omitted.	26
c_i	- the opportunity cost to the corporation as of the remainder due date of having paid $1 at date *i*. Where rates are stochastic, a superscript ω denotes the rate in a state of nature. In the single instalment model in chapter 3, the subscript is omitted.	26
e_j	- a partition of the loss $l(p;x)$	54
$f_{\omega n}$	- An alternative notation for *U*, *O*, *Pen*, and *Stub* in state of nature ω	76
G_i	- the prescribed rate of interest for period *i*. Where rates are constant, and in the single instalment model in chapter 3, the subscript is omitted.	24
g_i	- the amount of instalment interest owing by the corporation on the remainder due date for a deficiency of payment of $1 arising at date *i*. Where rates are stochastic, a superscript ω denotes the rate in a state of nature. In the single instalment model in chapter 3, the subscript is omitted.	24

		First Used at Page
$l(p;x)$	- the corporation's loss for any time path of payments	17
$L(p)$	- the corporation's expected loss	42
METR	- the corporation's marginal effective tax rate under the instalment structure	136
N_i	- the number of days in period i	24
O	- the opportunity loss or gain through overpaying or underpaying in the instalment period	17
p	- the vector of contingent payments (where X is known, this is a vector of 12 payments). In the single instalment model in chapter 3, p is a scaler.	16
p_i	- the payment for month i	16
p_s	- the payment on the remainder due date, if any	16
$\bar{p}$	- a vector of contingent payments in which all payments prior to the last month of the fiscal year equal zero	83
p^{ω}	- the 12-element subvector of p which relates to a particular ω	73
Pen	- the penalty associated with substantial underpayment in the instalment period	17
$Prob_{\omega}$	- the probability at date 1 of state ω	73
$prob_{\omega}$	- the probability of ω being within a partition Ω_l	84
q_i	- the corporation's instalment liability in month i. In the single instalment model in chapter 3, the subscript in omitted.	22
r	- the refund from Revenue Canada, if any	16

		First Used at Page
Stub	- the opportunity loss from the delay past the remainder due date in receiving a refund from the government	17
$t_{pv}(p;x)$	- the present value of all payments	36
U	- instalment interest (interest owing under section 161 from underpaying in the instalment period)	17
X	- the corporation's tax liability for the year as defined for the purposes of instalment payments	39
x	- 1/12 of the corporation's tax liability for the year (a subscript ω denotes the state of nature)	20
$\phi_{i,j}$	- an event at date i and history j	98
Ω	- the set of all states of nature	70
Ω_l	- a partition of Ω on Pen_ω and I_ω	84
ω	- a state of nature	70
$\mathscr{F}$	- a partition of the set of all states of nature, Ω	71

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